A Theory of Revealed Indirect Preference*

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Abstract

A preference over menus is said to be an indirect preference if it is induced by a preference over the objects that make up those menus, i.e., a menu A is ranked over B whenever A contains an object that is preferred to every object in B. The basic question we address in this paper is the following: suppose an observer has partial information of an agent's ranking over certain menus; what necessary and sufficient conditions on those rankings guarantee the existence of a preference over objects that induces the observed menu rankings? Our basic result has a wide variety of applications. (1) It gives a characterization of rankings over prices that could be extended to a bona fide indirect utility function. (2) It leads to a generalization of Afriat's (1967) Theorem that allows for imperfectly observed choices. (3) It could be used to characterize observations that are consistent with a multiple preferences model. (4) It leads to a characterization of a model of choice generated by minimax regret.

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1 Introduction

This paper explores the structure of indirect preference. Given a set of alternatives X, we refer to nonempty subsets of X as menus. A preference over menus constitutes an *indirect preference* so long as there is a preference over the alternatives in X such that menu A is preferred to another menu B whenever A contains an object that is preferred to every object in B.

The study of indirect preferences has a long history in economic theory. Indeed, a basic question in consumer theory concerns the recovery of the direct utility function (defined on bundles of ℓ goods) from the indirect utility over the prices of those goods. In this case, plainly, a vector of prices (with income held fixed at some value) corresponds to a linear budget set, which is just a specific type of menu from the consumption space $X = \mathbb{R}^{\ell}_{+}$. It is well-known that the crucial property that makes it possible for a function defined over prices to be a bona fide indirect utility function is for it to be quasiconvex in prices (see Krishna and Sonnenschein (1990) and Jackson (1986)).

This question could be posed in a more general form that is not specific to the consumer theory context. Kreps (1979) considers *all* possible menus drawn from a set of alternatives X and shows that a preference over these menus constitutes an indirect preference if and only if it satisfies the following property: if an agent prefers menu A to B, then he is indifferent between A and $A \cup B$.[1] Tyson (2018) extends this result by characterizing indirect preference defined over a given *subcollection* of menus; note that Tyson's result is also related to the classical result on indirect utility functions since the latter induces a preference defined over all linear budget sets, which forms a subcollection of menus among all possible menus drawn from the space of consumption bundles $X = \mathbb{R}^{\ell}_{+}$.

In this paper, we consider an observer who has access to a finite collection of observations, $\mathcal{M} := \left\{(A^t, B^t)\right\}_{t \in T}$. At each observation t, the observer knows that the agent weakly prefers menu A^t to B^t; and for a (possibly empty) subcollection S of those observations, the observer knows that A^t is strictly preferred to B^t for each $t \in S$. We say that the data set $\mathcal{M}$ can be *rationalized* if the observed preference between each pair of menus is part of an indirect preference, i.e., there exists a preference over the space of alternatives X such that A^t is (strictly) preferred to B^t if and only if A^t contains an alternative that is (strictly) better than each alternative

[1] Kreps (1979) uses this as a benchmark for the axiomatization of preference for flexibility. Other models of menu preferences include (among others) Dekel et al. (2001), Gul and Pesendorfer (2001), and Dekel and Lipman (2012).

in B^t.

Our fundamental result establishes that a data set $\mathcal{M}$ can be rationalized if and only if it satisfies the *never-covered property*; this is an intuitive property that could be defined via an iterative procedure. As a simple example of what it entails, suppose the observer knows that the agent strictly prefers menu A to B and weakly prefers A' to B'. Since there is an alternative in A that strictly dominates everything in B, a necessary condition for rationalization is that B does not contain A, i.e., $A \setminus B \neq \emptyset$. But this is not all. We also need to check if $A' \subseteq B$; if this holds, then clearly there is some element in A that dominates every element in $B \cup B'$ and so $A \setminus (B \cup B')$ must also be nonempty. In other words, some element must remain in A after all of A's revealed dominated elements have been iteratively excluded; this property turns out to be both necessary and sufficient for rationalization.

Note that the setting of our result is different from that of Kreps (1979), Tyson (2018), or the characterization results on indirect utility over prices. In those papers, it is assumed that the observer knows the *complete* ranking over the collection of menus under consideration. On the other hand, our result allows the observer to have only an incomplete ranking over menus; for example, the observer need not know the agent's preference between menus A^t and $A^{t'}$. The never-covered property reduces to the properties found in the earlier papers if the set of observations leads to a complete ranking over the collection of menus being considered.

Another paper that is closely related to ours is that of Fishburn (1976). That paper studies a different rationalizability problem (which is closely connected to the issue of coarse rationalizability discussed below), but its basic result could also be interpreted as a result on preference over menus; interpreted in that way, it provides a characterization of indirect preference (through a property called the *partial congruence axiom*) in the special case where A^t is *strictly* preferred to B^t for each $t \in T$. Our never-covered property is a generalization of the partial congruence axiom that allows for the possibility that the observer only knows that one menu is weakly preferred to another. This generalization is nontrivial and it is also crucial in certain applications, including the application to coarse rationalizability discussed below.

In some applications, it may not suffice to have a preference on X that rationalizes a data set $\mathcal{M}$; it may also be desirable to have the preference be the extension of some given preorder. For example, in the case of consumer theory, it would be natural to require any rationalizing preference to be increasing in the product order on the consumption space $X = \mathbb{R}^\ell_+$. In cases where the space of alternatives X, and the menus defined on it, contain infinitely many elements, it is also natural to assume that there is a topology on X and to require preferences over X to be continuous, which guarantees the existence of optimal elements when menus are compact. We show that our basic result can be extended to incorporate these features. Last but not least, we show that the never-covered property can be verified via an efficient algorithm, which facilitates the empirical application of our results.

The paper also discusses four applications of our theory.

(1) First, we revisit the question of characterizing indirect utility over prices. Instead of assuming that the entire indirect utility function is known, we assume that the observer only knows the consumer's preference for a finite set of price pairs, i.e., p^t is preferred to q^t (for $t = 1, 2, \ldots, T$), with income normalized at 1. We show that there is an increasing, continuous and concave utility function that rationalizes the observed price preferences provided the latter satisfies a generalization of the quasi-convex property (on indirect utility functions). In the case where p^t is strictly preferred to q^t for all t, the condition says the following: for any nonempty subset T' of observations, there exists $p^{t'}$ for some $t' \in T'$ such that $p^{t'}$ is no greater than any q in the convex hull of $\{q^s\}_{s \in T'}$.

(2) Consider a data set with T observations; at each observation t, it is observed that an agent chooses x^t from the menu C^t. Various versions of Afriat's theorem (1967)[2] answer the following question: what necessary and sufficient conditions on $\mathcal{O} = \{(x^t, C^t)\}_{t \in T}$ guarantee the existence of a utility function U defined on X such that $x^t \in \arg\max_{x \in C^t} U(x)$ for all observations t. Afriat's Theorem assumes that an observer knows precisely the choice made by the agent, but there are many scenarios where the observation is *coarse*, in the sense that the agent's choice cannot be precisely pinned down to x^t, but is known simply to come from a subset of C^t. For instance, suppose that goods are categorized into several categories, and the observer only knows the consumer's total spending on each category of goods at each observation. In this case, the observer will be able to infer that the consumer is choosing from a set $A^t \subseteq C^t$ but not the precise choice itself. Another possibility is that the observations are known to be imperfect; even though y^t is the recorded choice, one may then wish to have a more permissive check of rationalization where the true choice is allowed to deviate from y^t in a limited way.

These scenarios are formally captured by specifying an observation as (A^t, C^t), where $A^t \subseteq C^t$ contains the agent's choice. The corresponding rationalization

[2] See, for example, Varian (1982), Forges and Minelli (2009), Reny (2015), and Nishimura et al. (2017).

condition requires that there be a utility function U and x^t such that $x^t \in A^t$ and $x^t \in \arg\max_{x \in C^t} U(x)$ for all observations t. Notice that this is equivalent to the condition that (considered as menus) A^t is weakly preferred to C^t for all t. It follows that the never-covered property could be used to characterize rationalizable data sets and our algorithm provides a practical way of checking for this property in empirical applications.[3]

The last two applications of our theory provide characterizations of choice models outside the classical paradigm.

(3) In the multiple preferences model (Aizerman and Malishevski, 1981; Salant and Rubinstein, 2008), an agent's choice from a menu C could be the optimal choice for any one of a *set* of preferences. The question we pose is the following: suppose that for a finite collection of menus C^t we observe the agent's choices from each menu, which we denote by A^t; when can we find a collection of preferences $\{\succsim_i\}_{i \in I}$ such that $Z^t = A^t$, where Z^t consists of all elements of C^t that are optimal according to some preference $\succsim'$ in $\{\succsim_i\}_{i \in I}$? It turns out that this problem can be reformulated as a problem of rationalizing menu preferences and thus could be solved with a version of the never-covered property. Our result generalizes the finding in Aizerman and Malishevski (1981), who address this issue in the case where the choice set from *every* possible menu is observed.

(4) The second model we consider is the minimax regret model (Wald, 1950; Savage, 1951). In this model, an agent has multiple utility functions over alternatives, drawn from a set $\mathcal{U}$. For a given menu C, the regret of alternative x under one of the agent's utility functions $u \in \mathcal{U}$ is given by $u(x) - \max_{y \in A} u(y)$. The agent evaluates alternative x according to its maximal regret $\max_{u \in \mathcal{U}} (u(x) - \max_{y \in A} u(y))$, and chooses alternatives from the choice set that minimize the maximal regret. Using the never-covered property once again, we could find necessary and sufficient conditions on a data set under which the set of choices at each observed menu coincides exactly with the set of model consistent choices, for an appropriately chosen $\mathcal{U}$.

The rest of the paper is organized as follows. In Section 2, we set out the notation that are used throughout the paper. In Section 3, we introduce the main theorem, discuss several of its special cases, and also formulate the algorithm to

[3] Fishburn (1976) considers a related problem where he requires $\arg\max_{x \in C^t} U(x) \subseteq A^t$ for all observations t. This is equivalent to requiring the menu A^t to be strictly preferred to $C^t \setminus A^t$ for all t. We think that our formulation of rationalizability is more appropriate in empirical applications; unlike Fishburn's formulation, we allow for the possibility that an alternative in $C^t \setminus A^t$ is also optimal for the agent and it is the natural generalization of the rationalizability notion in Afriat's Theorem, which does allow for the optimality of alternatives in $C^t \setminus \{x^t\}$.

test the never-covered property. The four applications of our theory are presented (respectively) in Sections 4, 5, 6, and 7. All omitted proofs are in the Appendix.

2 Preliminaries

We work with a fixed nonempty set X, which can be viewed as the universal set of alternatives. Let $\mathcal{X}$ denote the collection of nonempty subsets of X. We refer to elements of $\mathcal{X}$ as menus. Generic elements of X are denoted by x, y, z, etc, while generic elements of $\mathcal{X}$ are denoted by A, B, C, etc.

A preorder $\trianglerighteq$ on X is a binary relation on X that is reflexive and transitive.[4] We use $\triangleright$ to denote the asymmetric part of $\trianglerighteq$. For a given preorder $\trianglerighteq$ on X and a set $A \in \mathcal{X}$, we define $A^{\downarrow}$ to be the decreasing closure of A with respect to the preorder $\trianglerighteq$, i.e.,

$$A^{\downarrow} := \{x \in X : y \trianglerighteq x \text{ for some } y \in A\},$$

and define $A^{\downarrow\downarrow}$ to be the strictly decreasing closure of A with respect to the preorder $\trianglerighteq$, i.e.,

$$A^{\downarrow\downarrow} := \{x \in X : y \triangleright x \text{ for some } y \in A\}.$$

We write $\max(A; \trianglerighteq) := A \setminus A^{\downarrow\downarrow}$ to denote the set of $\trianglerighteq$-undominated alternatives in A.

A *preference* $\succsim$ on X is a complete preorder on X. We use $\succ$ to denote the asymmetric part of $\succsim$. If $\succeq$ is anti-symmetric, then we call $\succ$ a *strict preference*. We say that x is a $\succsim$-maximal element in A if $x \in A$ and $x \succsim y$ for all $y \in A$, and write $\max(A; \succsim)$ to denote the set of $\succsim$-maximal alternatives in A. When convenient, we write $x \succsim A$ if $x \succsim y$ for each $y \in A$ and $x \succ A$ if $x \succ y$ for each $y \in A$.

Oftentimes, it is useful to study preferences restricted to a particular class. We say that the preference $\succsim$ extends the preorder $\trianglerighteq$ if

$$x \succsim y \text{ whenever } x \trianglerighteq y, \text{ and } x \succ y \text{ whenever } x \triangleright y.$$

We can think of $\trianglerighteq$ as an exogenously given dominance relation on X, and view the statement $x \trianglerighteq y$ as saying that x is an objectively better alternative than y, which the individual's preference $\succsim$ should respect.

[4] Terminology: a binary relation R on X is a nonempty subset of $X \times X$, but as usual, we write xRy instead of $(x, y) \in R$. We say that R is reflexive if xRx for each $x \in X$, transitive if $xRyRz$ implies xRz for each $x, y, z \in X$, complete if either xRy or yRx holds for any $x, y \in X$, and anti-symmetric if for any $x, y \in X$ such that $x \neq y$, xRy and yRx do not hold simultaneously. The asymmetric part of R is defined as the binary relation P on X such that xPy if and only if xRy but not yRx.

The general choice environment we have defined here broadly follows that in Nishimura et al. (2017). As a basic example of this environment, we note that in consumer theory, the consumption space with n commodities is typically $X = \mathbb{R}^n_+$ and it is common to assume that a consumer would always strictly prefer to have more of any good. In this case, the consumer's preference would extend the coordinatewise ordering $\geq$ on $\mathbb{R}^n_+$.[5] Bear in mind that this setup allows for an arbitrary preference without a preorder restriction, since this can be thought of as the case in which the preference extends the trivial preorder $\trianglerighteq$ where $x \trianglerighteq y$ if and only if $x = y$.

3 Rationalizability of menu preferences

In this section, we study the conditions under which a finite list of observed menu preference pairs collected from an agent is consistent with some (unobserved) preference on the underlying alternatives. The data collected by the observer is formally represented as $\mathcal{M} := \left\{(A^t, B^t)\right\}_{t\in T}$, where T is a nonempty finite index set and A^t and B^t are menus. For each t, the observer either knows that the agent weakly prefers A^t to B^t or that the agent strictly prefers A^t to B^t.[6] Let W be the collection of observations where A^t is weakly preferred to B^t, and let S be the collection of observations where A^t is strictly preferred to B^t. By definition, $\{W, S\}$ is a partition of T. For any nonempty $T' \subseteq T$, we let

$$A(T') := \textstyle\bigcup_{t\in T'} A^t \text{ and } B(T') := \bigcup_{t\in T'} B^t.$$

The following definition specifies precisely what it means for $\mathcal{M}$ to be rationalized.

Definition 1. *A set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ is rationalized by a preference $\succsim$ on X if for any $t \in T$, there exists $x^t \in A^t$ such that*

(1) $x^t \succsim B^t$ and

(2) $x^t \succ B^t$ if $t \in S$.

In this case, we say that $\mathcal{M}$ is rationalizable*. We say that a preference $\succsim$* $\trianglerighteq$-rationalizes *$\mathcal{M}$ if $\succsim$ rationalizes $\mathcal{M}$ and extends $\trianglerighteq$; in this case, $\mathcal{M}$ is* $\trianglerighteq$-

[5] Formally, $x \geq y$ if and only if $x_i \geq y_i$ for each $i \in \{1, 2, \ldots, n\}$, $x > y$ if and only if $x \geq y$ and $x_i > y_i$ for some $i \in \{1, 2, \ldots, n\}$. We write $x \gg y$ if and only if $x_i > y_i$ for each $i \in \{1, 2, \ldots, n\}$.

[6] This formulation includes the case in which the observer knows that the agent is indifferent between two menus (say) A and B, because this case could be considered as two observations, with the agent weakly preferring A to B in one observation and B to A in the other observation.

rationalizable.[7]

Our objective in this section is to characterize those sets of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ which can be $\trianglerighteq$-rationalized. Readers familiar with the revealed preference theory will notice that the issue is fairly straightforward when A^t is a singleton for each t. In that case, the problem is (in its essentials) within the scope of the well-known theorems of Afriat (1967) and Richter (1966) and their extensions;[8] in the manner of those theorems, some version of a no-cycling condition on the revealed preference relations defined on $\{x^t\}_{t\in T}$ (which will be stated formally later in Definition 3) is both necessary and sufficient for the existence of a preference that extends $\trianglerighteq$ and satisfies (1) and (2) in Definition 1.[9] Of course, A^t is typically not a singleton. Thus, we could understand the issue before us in the following way: we have to formulate a property on $\mathcal{M}$ that guarantees *the existence of a selection* x^t from A^t, such that the resulting (notional) set of observations $\left\{(\{x^t\}, B^t)\right\}_{t\in T}$ satisfies the required no-cycling condition. The next subsection provides the property guaranteeing that such a selection exists.

3.1 The never-covered property

Suppose that the data set $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ is $\trianglerighteq$-rationalized by some preference relation $\succsim$. Consider an arbitrary observation t. By the definition of $\trianglerighteq$-rationalizability, A^t contains an alternative x with $x \succsim B^t$. Since $\succsim$ extends $\trianglerighteq$, $x \notin B^{t\downarrow\downarrow}$. Moreover, if $t \in S$, then A^t contains an alternative x with $x \succ B^t$ and so $x \notin B^{t\downarrow}$. Thus A^t cannot be covered by $B^{t\downarrow\downarrow}$ if $t \in W$, and cannot be covered by $B^{t\downarrow}$ if $t \in S$.

[7] Definition 1 is one of several possible formulations of the rationalizability of a set of menu preference pairs. For example, one could require that for any $t \in T$, (1) for any $y \in B^t$, there exists $x \in A^t$ such that $x \succeq y$ and (2) there exists $x^t \in A^t$ such that $x^t \succ B^t$ if $t \in S$. To see the (subtle) differences between these two formulations, suppose that the preorder is $\geq$. The first data set has only one observation (B, B) where $B = (0, 1)$ and the relation is weak. The second data set has only one observation (A, B) where A is the set of rational numbers in $(0, 1)$, $B = (0, 1)$, and the relation is weak. Definition 1 would classify both data sets as not $\geq$-rationalizable, while the alternative formulation would classify both as $\geq$-rationalizable. That being said, in all the applications that we study (and in most economic environments), either X is fine or X is infinite but the menus are compact. In these cases, these two formulations are equivalent.

[8] For extensions of Afriat's Theorem, see, for example, Varian (1982), Forges and Minelli (2009), Reny (2015), and Nishimura et al. (2017).

[9] To be precise, Nishimura et al. (2017) already provides a condition on $\left\{(\{x^t\}, B^t)\right\}_{t\in T}$ that is necessary and sufficient for the existence of a preference $\succsim$ that extends $\trianglerighteq$ and satisfies $x^t \succsim B^t$ for each $t \in T$. In our case, we potentially have observations where we require $x^t \succ B^t$; thus a modification of the condition in Nishimura et al. (2017) is required to accommodate these cases, but this extension of their result is fairly straightforward.

This argument could be generalized to more than one observation. For any nonempty subset $T' \subseteq T$, notice that (1) if x satisfies $x \succsim B(T')$ then $x \notin B(T')^{\downarrow\downarrow}$; and (2) if x satisfies $x \succ B(T' \cap S)$ then $x \notin B(T' \cap S)^{\downarrow}$. Thus, if $A(T')$ contains an alternative $\hat{x}$ satisfying both conditions, then $A(T')$ cannot be covered by $B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}$. Indeed, we can find such an alternative $\hat{x}$ in $A(T')$: for each $t \in T'$, pick $x^t \in A^t$ such that $x^t \succsim B^t$ if $t \in W$ and $x^t \succ B^t$ if $t \in S$; then let $\hat{x} = \max(\{x^t\}_{t \in T'}; \succsim) \in A(T')$.

We are now ready to introduce the procedure that we call *the iterated exclusion of dominated observations*. Given a nonempty subset of observations T', let $\Phi^0(T') := \emptyset$, and let $\Phi^1(T')$ be the collection of observations t such that A^t is covered by $B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}$, i.e.,

$$\Phi^1(T') := \left\{ t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow} \right\}.$$

We know there is $\hat{x} \in A(T')$ such that $\hat{x} \succsim B(T')$ and $\hat{x} \succ B(T' \cap S)$. Then, $\hat{x} \succ A^t$ for each $t \in \Phi^1(T')$ and, since A^t is preferred to B^t, we obtain that $\hat{x} \succ B^{t\downarrow}$ for each $t \in \Phi^1(T')$. Thus, $\hat{x} \succ B((T' \cap S) \cup \Phi^1(T'))^{\downarrow}$. Obviously, $A(T')$ cannot be covered by $B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^1(T'))^{\downarrow}$. Let

$$\Phi^2(T') := \left\{ t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^1(T'))^{\downarrow} \right\}.$$

Note that $\Phi^1(T') \subseteq \Phi^2(T')$. By a similar argument, we know that if $t \in \Phi^2(T')$ then $\hat{x} \succ B^{t\downarrow}$ and consequently, $A(T')$ cannot be covered by $B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^2(T'))^{\downarrow}$. We may repeat this argument for $m = 2, 3, \ldots$, where

$$\Phi^{m+1}(T') := \left\{ t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^m(T'))^{\downarrow} \right\}.$$

Since $\Phi^m(T')$ is an increasing sequence in m in the set inclusion sense and the data set is finite, the procedure stops at m^* when $\Phi^{m^*}(T') = \Phi^{m^*+1}(T')$. Let $\Phi(T') := \Phi^{m^*}(T')$; we refer to $\Phi(T')$ as the set of *revealed dominated observations* (or simply *dominated observations*) in T'. Since $B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi(T'))^{\downarrow}$ cannot contain $\hat{x}$, it does not cover $A(T')$. In other words, $\Phi(T')$ is a strict subset of T'.

Definition 2. $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ *satisfies the never-covered property under* $\trianglerighteq$ *if, for any nonempty* $T' \subseteq T$*, the set of revealed dominated observations* $\Phi(T')$ *satisfies* $\Phi(T') \neq T'$.

We have shown that the never-covered property under $\trianglerighteq$ is a necessary condition for a data set to be $\trianglerighteq$-rationalizable. The main result of this paper, Theorem 1,

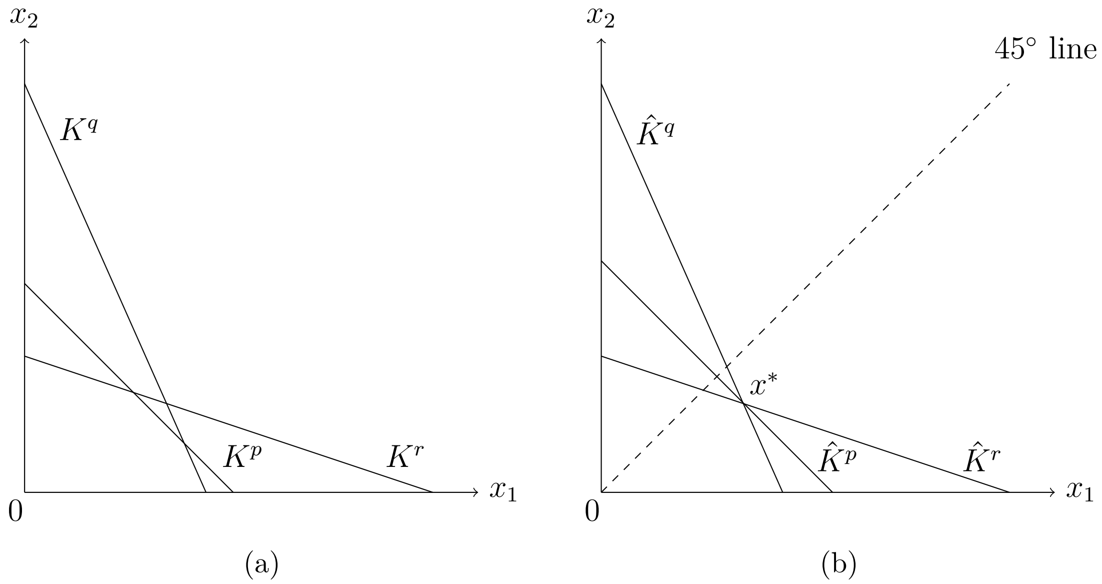

Figure 1: (a) The data set $\mathcal{M} = \{(K^p, K^q)), (K^q, K^r)\}$, where both relations are weak, is not $\geq$-rationalizable; (b) the data set $\hat{\mathcal{M}} = \{(\hat{K}^p, \hat{K}^q)), (\hat{K}^q, \hat{K}^r)\}$, where both relations are weak, is $\geq$-rationalizable.

shows that it is also sufficient. The example below illustrates how we can use the never-covered property under $\trianglerighteq$ to test the $\trianglerighteq$-rationalizability of a set of menu preference pairs.

Example 1. Consider the classic model of consumer demand with two goods. We take the preorder to be the coordinatewise ordering $\geq$ on $X = \mathbb{R}^2_+$. Figure 1(a) depicts linear budget sets, K^p, K^q, and K^r. Figure 1(b) depicts linear budget sets, $\hat{K}^p$, $\hat{K}^q$, and $\hat{K}^r$.[10]

Suppose that $\mathcal{M}$ consists of two observations, (K^p, K^q) and (K^q, K^r), where both relations are weak. We claim that $\mathcal{M}$ is not $\geq$-rationalizable. Suppose to the contrary that $\mathcal{M}$ is $\geq$-rationalizable, say, by a preference relation $\succsim$. Then, there exists at least one bundle $\hat{x}$ contained in $K^p \cup K^q$ such that $\hat{x} \succsim K^q \cup K^r$. Since $K^q \cup K^r = K^p \cup K^q \cup K^r$, the bundle $\hat{x}$ is $\succsim$-maximal in $K^p \cup K^q \cup K^r$. However, any $\succsim$-maximal alternative in $K^p \cup K^q \cup K^r$ cannot be in K^p since (i) $\succsim$ extends $\geq$; and (ii) K^p is covered by $(K^q \cup K^r)^o$.[11] Furthermore, $\hat{x}$ cannot be contained in K^q either; otherwise, there would be some $y \in K^p$ such that $y \succsim \hat{x}$ and thus y is $\succsim$-maximal in $K^p \cup K^q \cup K^r$, which is a contradiction. Notice that our argument corresponds precisely to a violation of the never-covered property under $\geq$ for $T' = T$. Since $B(T)^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow} = (K^q \cup K^r)^o$ covers K^p, $\Phi^1(T)$ contains the

[10] The 45 degree line in Figure 1(b) will be used in Example 4.

[11] For any set K, we use K^o to denote its interior.

observation (K^p, K^q). Since $K^q \subseteq B(\Phi^1(T))$, $K^q \subseteq B(T)^{\downarrow\downarrow} \bigcup B((T \cap S) \cup \Phi^1(T))^{\downarrow}$ and $\Phi^2(T) = T$. Thus, $\Phi(T) = T$.

On the other hand, the data set $\hat{\mathcal{M}} = \{(\hat{K}^p, \hat{K}^q)), (\hat{K}^q, \hat{K}^r)\}$, where both relations are weak, is $\geq$-rationalizable. The optimal bundle in each set must be x^*, and it is easy to check that the set $\Phi(T)$ is empty.

3.2 The basic result

Our proof of the sufficiency of the never-covered property under $\trianglerighteq$ in guaranteeing the $\trianglerighteq$-rationalizability of $\mathcal{M}$ proceeds by explicitly providing a way of selecting x^t in A^t for each t such that there exists a preference $\succsim$ on X that extends $\trianglerighteq$ and satisfies (1) $x^t \succsim B^t$ for all $t \in T$ and (2) $x^t \succ B^t$ for $t \in S$ (see Definition 1). Suppose that we have selected x^t from A^t for each t in some way. How do we check whether there exists a preference with the required conditions? This can be characterized by a no-cycling property which we now explain.

Let $Y = \{x^t\}_{t \in T}$. For x^t and $x^{t'}$ in Y, we say that x^t is *revealed preferred to* $x^{t'}$ and denote it by $x^t \, R \, x^{t'}$ if $x^{t'} \in B^{t\downarrow}$, and we say that x^t is *revealed strictly preferred to* $x^{t'}$ and denote it by $x^t \, P \, x^{t'}$ if either (i) $x^{t'} \in B^{t\downarrow\downarrow}$ or (ii) $t \in S$ and $x^{t'} \in B^{t\downarrow}$. The following is a no-cycling condition on the binary relations R and P.

Definition 3. *Given a data set $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$, a selection x^t from A^t for each $t \in T$ is a no-cycling selection if the revealed preference relations R and P obey the following no-cycling property: there does not exist x^{t_1}, $x^{t_2} \ldots, x^{t_n}$ in $Y = \{x^t\}_{t \in T}$ such that*

$$x^{t_1} \, R \, x^{t_2} \, R \cdots R \, x^{t_n} \text{ and } x^{t_n} \, P \, x^{t_1}. \tag{1}$$

To see why this could be a plausible characterization, note that it is plainly a *necessary* condition. Indeed, suppose the preference $\succsim$ extends $\trianglerighteq$ and, with this preference, x^t satisfies (1) $x^t \succsim B^t$ for all $t \in T$ and (2) $x^t \succ B^t$ for $t \in S$. If x^t is revealed preferred to $x^{t'}$, then by definition, $x^t \succsim y \trianglerighteq x^{t'}$ for some $y \in B^t$; since $\succsim$ extends $\trianglerighteq$ and $\succsim$ is transitive, we obtain $x^t \succsim x^{t'}$. If x^t is revealed strictly preferred to $x^{t'}$, then we have either (i) $x^t \succsim y \triangleright x^{t'}$ for some $y \in B^t$ or (ii) $t \in S$ and $x^t \succ y \trianglerighteq x^{t'}$ for some $y \in B^t$; in both cases, we may conclude that $x^t \succ x^{t'}$. Given the transitivity of $\succsim$, we plainly cannot have x^{t_1}, $x^{t_2} \ldots, x^{t_n}$ in Y satisfying (1).

We have just shown that if a data set of menu preference pairs is $\trianglerighteq$-rationalizable, then it admits a no-cycling selection. Theorem 1 below states that the converse is also true and that both are equivalent to the never-covered property under $\trianglerighteq$.

Theorem 1. *Given a set of menu preference pairs* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ *and a preorder* $\trianglerighteq$*, the following statements are equivalent:*

1. $\mathcal{M}$ *is* $\trianglerighteq$*-rationalizable.*
2. $\mathcal{M}$ *satisfies the never-covered property under* $\trianglerighteq$*.*
3. $\mathcal{M}$ *admits a no-cycling selection.*

We end this subsection by providing a different formulation of the never-covered property under $\trianglerighteq$. The never-covered property under $\trianglerighteq$ implies that for any nonempty $T' \subseteq T$, there exists some $\Phi \subsetneq T'$ such that for any $t \in T' \setminus \Phi$, $A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi)^{\downarrow}$. To see this, we could simply set $\Phi = \Phi(T')$. The following proposition shows that the converse is also true.

Proposition 1. *Consider the data set* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$.

1. *If for some nonempty* $T' \subseteq T$*, there exists* $\Phi \subsetneq T'$ *such that for any* $t \in T' \setminus \Phi$*,*
$$A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi)^{\downarrow},$$
then $\Phi(T') \subseteq \Phi$*.*
2. *If for any nonempty* $T' \subseteq T$*, there exists* $\Phi \subsetneq T'$ *such that for any* $t \in T' \setminus \Phi$*,*
$$A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi)^{\downarrow},$$
then $\mathcal{M}$ *satisfies the never-covered property under* $\trianglerighteq$*.*

3.3 Algorithm

Given a subset T', it is straightforward to check whether $\Phi(T') = T'$. Thus, Theorem 1 provides us with a way of checking if a set of menu preference pairs is $\trianglerighteq$-rationalizable: we need to check whether $\Phi(T') \neq T'$ for all $T' \subseteq T$. This may not seem promising as an empirical procedure, since for a data set with n observations, we would have to go through all $2^n - 1$ nonempty subsets of T to guarantee the $\trianglerighteq$-rationalizability of that set. In this subsection, we provide a simple algorithm to check whether the never-covered property under $\trianglerighteq$ holds. This algorithm requires us to check whether $\Phi(T') \neq T'$ for at most n subsets of T. Thus, the never-covered property under $\trianglerighteq$ can be checked in an efficient manner.

Following the convention in the computer science literature, we use k' to denote the updated value of a variable k.

Algorithm. Set $T^0 := T$. Set $k := 1$.

START. Derive $T^k := \Phi(T^{k-1})$. Consider the following three mutually exclusive cases:

(a). $T^k = \emptyset$.

Stop and output $\trianglerighteq$-*Rationalizable*.

(b). $\emptyset \neq T^k \subsetneq T^{k-1}$.

Go to START with $k' = k + 1$.

(c). $\emptyset \neq T^k = T^{k-1}$.

Stop and output *Not* $\trianglerighteq$-*Rationalizable*.

Note that the ALGORITHM is effectively checking whether $\Phi(T^k) = T^k$ for an endogenous sequence of subsets of T. We emphasize that, for a data set with n observations, the ALGORITHM necessarily terminates within n steps, and we only need to check at most n subsets of T.

Proposition 2 below provides the justification for this ALGORITHM.

Proposition 2. *The set of menu preference pairs* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ *is* $\trianglerighteq$-*rationalizable if and only if the* ALGORITHM *outputs* $\trianglerighteq$-Rationalizable*.*

3.4 Nice rationalization when $\trianglerighteq$ is trivial

The analysis in the previous subsections holds for any preorder $\trianglerighteq$ and any partition $\{W, S\}$ of T. In this subsection, we focus on the important special case in which the rationalizing preference is not required to extend any given preorder or, put another way, the preorder $\trianglerighteq$ is simply the trivial preorder where $x \trianglerighteq y$ if and only if $x = y$.

Consider a set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ where S is nonempty.[12] Since $\trianglerighteq$ is trivial, $A^{\downarrow} = A$ and $A^{\downarrow\downarrow} = \emptyset$ for all A. Thus, the procedure of iterated exclusion of dominated observations reduces to the following: for any nonempty $T' \subseteq T$,

$$\Phi^1(T') = \left\{t \in T' : A^t \subseteq B(T' \cap S)\right\} \text{ and}$$
$$\Phi^{m+1}(T') = \left\{t \in T' : A^t \subseteq B\left((T' \cap S) \cup \Phi^m(T')\right)\right\}, \text{ for } m = 1, 2, \ldots.$$

This iteration must stop at some point, i.e., there is m^* such that $\Phi^{m^*}(T') = \Phi^{m^*+1}(T')$. The set of dominated observations is $\Phi(T') := \Phi^{m^*}(T')$. By definition, $\mathcal{M}$ satisfies the never-covered property under the trivial preorder if $\Phi(T') \neq T'$

[12] If $\trianglerighteq$ is trivial, then any data set such that $S = \emptyset$ is rationalizable (by the preference relation that the DM is indifferent among all alternatives).

for all nonempty $T' \subseteq T$. For the sake of brevity, we would simply refer to $\mathcal{M}$ as *satisfying the never-covered property*, if it satisfies the never-covered property under the trivial preorder.

The following example is a simple illustration of the application of Theorem 1 and the ALGORITHM to this setting.

Example 2. Let $X = \{x, y, z, r, w\}$ and suppose that $\trianglerighteq$ is trivial. The data set $\mathcal{M}$ consists of the following observations:

$$\begin{aligned}
&A^1 = \{x, y\},\ B^1 = \{z, r, w\};\\
&A^2 = \{y, z\},\ B^2 = \{x, r, w\};\\
&A^3 = \{x, r\},\ B^3 = \{y, z, w\};\ \textit{and}\\
&A^4 = \{r, w\},\ B^4 = \{z\}.
\end{aligned}$$

where A^1 is strictly preferred to B^1 and the other relations are weak. In this case, $T = \{1, 2, 3, 4\}$ and $S = \{1\}$. Since $A^4 \subseteq B^1$, we obtain $\Phi^1(T) = \{4\}$. Since $B^1 \cup B^4 = B^1$, $\Phi^2(T) = \{4\}$, and thus $\Phi(T) = \{4\}$. The ALGORITHM now directs us to calculate $\Phi(\{4\})$; this set is empty and so the ALGORITHM concludes that $\mathcal{M}$ is rationalizable. And indeed it is: for example, the preference $x \sim y \succ r \sim w \sim z$ rationalizes $\mathcal{M}$.

The following definition imposes a stronger notion of rationalizability than the one provided in Definition 1 because the preference $\succsim$ is required to have an optimum in every menu in the list of observations.

Definition 4. *A set of menu preference pairs* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ *is nicely rationalized by a preference* $\succsim$ *on* X *if for all* $t \in T$, $\max(A^t; \succsim)$ *and* $\max(B^t; \succsim)$ *are nonempty, and for any* $x^t \in \max(A^t; \succsim)$ *and* $y^t \in \max(B^t; \succsim)$, *we have (1)* $x^t \succsim y^t$ *and (2)* $x^t \succ y^t$ *if* $t \in S$. *In this case, we say that* $\mathcal{M}$ *is nicely rationalizable. A preference* $\succsim$ nicely $\trianglerighteq$-rationalizes $\mathcal{M}$ *if* $\succsim$ *nicely rationalizes* $\mathcal{M}$ *and extends* $\trianglerighteq$*; in this case, we say that* $\mathcal{M}$ *is* nicely $\trianglerighteq$-rationalizable.

In general, it is possible for a preference to rationalize a data set without it being a nice rationalization. Of course, this cannot happen when X is finite, since the existence of an optimum given a preference is then guaranteed. The next result says that it cannot happen when the preorder is trivial either, in the sense that every data set that is rationalizable (by some preference) is also nicely rationalizable (by a possibly different preference).

Theorem 2. *The following statements on the data set* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ *are equivalent:*

1. *$\mathcal{M}$ is rationalizable.*
2. *$\mathcal{M}$ satisfies the never-covered property.*
3. *$\mathcal{M}$ is nicely rationalizable.*

Example 3. As an illustration of Theorem 2, consider the case in which the data set consists of just one observation: $A^1 = \{1\}$, $B^1 = (0, 1)$, with A^1 strictly preferred to B^1. Can this data set be rationalized by a preference that extends the standard total order $\geq$ on $\mathbb{R}$? Clearly, such a rationalization exists; in fact, $\geq$ is itself the unique rationalization. However, this is not a nice rationalization since $(0, 1)$ does not have an optimum according to $\geq$. On the other hand, Theorem 2 guarantees that there *is* a nice rationalization of the strict preference of A^1 over B^1 if we do not require the rationalizing preference to extend $\geq$. And indeed it does: simply let $1 \succ r$ for all $r < 1$, and for all $r,\ r' \in (0, 1)$ let $r \sim r'$.

3.5 Strict menu preferences and strict rationalization

We now turn to the case in which $\trianglerighteq$ is trivial and $T = S$, so that the rationalizability of a data set $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ reduces to the following: there exists a preference $\succsim$ on X such that for each t, there exists $x \in A^t$ with $x \succ B^t$. This case is of particular interest, as the procedure of iterated exclusion of dominated observations ends in one round and the never-covered property has a much simpler form. To wit, since $T = S$, for any nonempty $T' \subseteq T$, $B(T' \cap S) = B(T')$. Therefore, the procedure of iterated exclusion of dominated observations reduces to the following: for any nonempty $T' \subseteq T$,

$$\Phi^1(T') = \left\{t \in T' : A^t \subseteq B(T' \cap S)\right\} = \left\{t \in T' : A^t \subseteq B(T')\right\} \text{ and}$$
$$\Phi^2(T') = \left\{t \in T' : A^t \subseteq B((T' \cap S) \cup \Phi^1(T'))\right\} = \left\{t \in T' : A^t \subseteq B(T')\right\} = \Phi^1(T').$$

Therefore, the set of dominated observations is $\Phi(T') = \{t \in T' : A^t \subseteq B(T')\}$, and the never-covered property $\Phi(T') \neq T'$ holds if and only if $A(T') \not\subseteq B(T')$.

We note that this special case of our result is (in its essentials) covered by Fishburn (1976), who establishes that there is $\succsim$ such that $x^t \in A^t$ with $x^t \succ B^t$ for each t if and only if

$$A(T') \not\subseteq B(T') \text{ for all nonempty } T' \subseteq T. \tag{2}$$

Following Fishburn, we shall refer to the property (2) as the *partial congruence axiom*. In fact, Fishburn's result is somewhat more general because it partially covers the case in which T is infinite.[13] We have confined our attention to the case in which T is finite because it is the case most relevant to empirical applications and it allows us to formulate an efficient algorithm for checking the never-covered property.[14]

The following result summarizes our findings when menu preferences are strict.

Corollary 1. *For a data set $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ where $T = S$, the following statements are equivalent:*

1. *$\mathcal{M}$ is nicely rationalizable.*
2. *$\mathcal{M}$ is nicely rationalizable by a strict preference.*
3. *$\mathcal{M}$ satisfies the partial congruence axiom.*

The equivalence of the first and second statements in this corollary is due to Fishburn (1976, Lemma 1). As we have explained, the partial congruence axiom and the never-covered property are equivalent when $T = S$ and thus the equivalence of the first and third statements follows from Theorem 2.

It almost goes without saying that when S is a strict subset of T (so that there are some observations where rationalization only requires $x^t \in A^t$ such that $x^t \succsim B^t$ rather than $x^t \succ B^t$), the partial congruence axiom no longer characterizes rationalizable data sets and one needs to appeal to the never-covered property. Indeed, consider the data set in Example 2; is it possible for $A^t \succ B^t$ for all t? The answer is 'No' because $A(T) = B(T)$ and the partial congruence axiom is violated. However, if we only require the relation in the first observation to be strict (as we did in that example), then the data set *is* rationalizable because it satisfies the (weaker) never-covered property.

We now discuss the relationship between our work and the paper of de Clippel and Rozen (forthcoming). We first describe the problem they solve using our terminology. They assume that the preorder $\trianglerighteq$ is trivial. They consider finite data

[13] Fishburn (1976) considers two separate cases. For the case in which T could be infinite but A^t is required to be *finite* for each t, he shows that rationalizability is characterized by the partial congruence axiom. When T is countable and A^t is allowed to be infinite, his characterization result (Theorem 3) takes a different form, but it is equivalent to the partial congruence axiom when T is finite. The case in which T is more than countable and A^t is infinite is not covered by his results (or ours).

[14] If T is infinite, then there is obviously no hope of any algorithm for checking rationalizability. Fishburn's paper (perhaps partly because of its emphasis on the case of infinite T) does not discuss algorithms for checking the partial congruence axiom.

sets where each observation t has the form $(\{A_j^t\}_{j \in J(t)}, x^t)$, where $\{A_j^t\}_{j \in J(t)}$ is a collection of subsets of X and develop an algorithm that enables them to determine if $\left\{(\{A_j^t\}_{j \in J(t)}, x^t)\right\}_{t \in T}$ admits an *upper contour rationalization* in the following sense: there is a strict preference $\succ$ such that, at each t, there is a set in the collection $\{A_j^t\}_{j \in J(t)}$ that is contained in the upper contour of x^t, i.e., there is $A_{j(t)}^t$ in $\{A_j^t\}_{j \in J(t)}$ such that $A_{j(t)}^t \succ x$.[15]

This problem and our menu rationalization problem have different economic motivations; however, when X is finite (so that all the relevant subsets in both problems are also finite), the two problems could be thought of as equivalent in the sense that it is always possible to convert one problem into the other, which also means that any algorithm developed for one could, in principle, be used to solve the other. That said, it should be clear from the conversion procedure (outlined in the Appendix) that there is no general computational reason for solving either problem in this roundabout fashion, since the converted data set would typically have more observations than the original data set. Thus, the two algorithms are best understood as distinct and serving different purposes.

3.6 Menu preferences in Kreps (1979) and Tyson (2018)

Let $\hat{\mathcal{X}} \subseteq \mathcal{X}$ be a nonempty collection of menus and let $\succsim^M$ be a preference over $\hat{\mathcal{X}}$ (which means that $\succsim^M$ is a reflexive, transitive, and complete binary relation on $\hat{\mathcal{X}}$). Abusing our terminology somewhat, we say that a preference $\succsim$ on X nicely rationalizes $\succsim_M$ if, for all $D \in \hat{\mathcal{X}}$, the set $\arg\max(D; \succsim)$ is nonempty and, for any $x' \in \arg\max(D'; \succsim)$ and $x'' \in \arg\max(D''; \succsim)$, we have $x' \succsim x''$ if $D' \succsim^M D''$ and $x' \succ x''$ if $D' \succ^M D''$.

Tyson (2018) shows that a menu preference $\succsim^M$ on $\hat{\mathcal{X}}$ is nicely rationalizable if and only if it satisfies the cover dominance condition, as defined below.

Definition 5 (Tyson (2018))**.** *A menu preference $\succsim^M$ over $\hat{\mathcal{X}}$ satisfies the cover dominance condition if for any $A, D \in \hat{\mathcal{X}}$ and $\{B_i\}_{i \in I} \subseteq \hat{\mathcal{X}}$,*

$$A \succ^M B_i \text{ for each } i \in I \text{ and } D \subseteq \cup_{i \in I} B_i \Rightarrow A \succ^M D.$$

Furthermore, when $\hat{\mathcal{X}}$ is finite and closed under union, the cover dominance condition can be equivalently stated in the following way.

[15] A version of their algorithm is contained in the first working paper version of their paper; see de Clippel and Rozen (2012).

Definition 6. *Let $\hat{\mathcal{X}}$ be closed under union. A menu preference $\succsim^M$ over $\hat{\mathcal{X}}$ is Kreps-consistent if, for any $A, B \in \mathcal{X}$, $A \succsim^M B$ implies that $A \sim^M A \cup B$.*[16]

Hence, Tyson (2018) generalizes the result in Kreps (1979) which states that, when X is finite, a menu preference on $\mathcal{X}$ (the set of all nonempty subsets of X) is nicely rationalizable if and only if it is Kreps-consistent.

In our setup, it is assumed that $\mathcal{M}$, a finite list of preference pairs between menus A^t and B^t, is observed. Its key difference with Kreps (1979) and Tyson (2018) is that these observations need not constitute a preference over all the menus in $\hat{\mathcal{X}} = \{A^t\}_{t \in T} \cup \{B^t\}_{t \in T}$; in other words, while we posit that it is known that A^t is weakly preferred to B^t, we do not require the observer to know how A^t compares with $A^{t'}$ or with $B^{t'}$.

Provided that the collection of menus $\hat{\mathcal{X}}$ is finite, we could always construct a finite set of menu preferences $\mathcal{M}^*$ from a preference $\succsim^M$ on $\hat{\mathcal{X}}$ in the following way:

$$(A^t, B^t) \in \mathcal{M}^* \text{ if and only if } A^t \succsim^M B^t \text{ and } t \in S \text{ if and only if } A^t \succ^M B^t.$$

Clearly, $\mathcal{M}^*$ is nicely rationalizable (in the sense of Definition 4) by a preference $\succsim$ if and only if $\succsim$ nicely rationalizes the menu preference $\succsim^M$. By Theorem 2, the nice rationalizability of $\succsim^M$ is characterized by the never-covered property on $\mathcal{M}^*$. In particular, this means that one can efficiently check if $\succsim^M$ is nicely rationalizable by implementing our algorithm for checking the never-covered property. It also follows immediately from Theorem 2 and the results in Tyson (2018) that, when $\hat{\mathcal{X}}$ is finite, the cover dominance condition on $\succsim^M$ and the never-covered property on $\mathcal{M}$ are equivalent; furthermore, if $\hat{\mathcal{X}}$ is also closed under union, then each of these conditions is equivalent to Kreps-consistency.

3.7 Continuous rationalizability

When the space of alternatives X is infinite, it is helpful to endow it with a topology and study continuous preferences. This guarantees (among other things) that the preference generates an optimum choice on compact menus and that the optimum varies continuously with the menu. For example, a continuous preference on the consumption space $\mathbb{R}^n_+$ would guarantee that the demand correspondence is nonempty when prices are strictly positive (so that the budget set is compact) and varies continuously with prices.[17]

[16] We denote by $\sim^M$ the equivalence relation induced from $\succsim^M$.

[17] More generally, if we endow the collection of compact menus with the Hausdorff metric, then the correspondence mapping a compact menu to its optima is well-defined and upper hemicontinous.

We say that a set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ is $\trianglerighteq$-rationalized by a continuous utility function $u : X \to \mathbb{R}$ if u represents a preference $\succsim$ that $\trianglerighteq$-rationalizes $\mathcal{M}$ in the sense of Definition 1. Theorem 3 below provides conditions under which $\mathcal{M}$ can be rationalized by a continuous utility function.

Theorem 3. *Suppose that X is a locally compact and separable metric space and $\trianglerighteq$ is a continuous preorder on X.*[18] *For the set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ where B^t is compact for each $t \in T$, the following statements are equivalent:*

1. *$\mathcal{M}$ is $\trianglerighteq$-rationalizable.*
2. *$\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$.*
3. *$\mathcal{M}$ is $\trianglerighteq$-rationalized by a continuous utility function u.*

Note that this theorem does not assume that A^t is a compact set. It does assume that B^t is a compact set, which guarantees that for any continuous utility function u, the set $\arg\max_{x\in B^t} u(x)$ is nonempty. If, in addition, A^t is a compact set for all t, then $\arg\max_{x\in A^t} u(x)$ is also nonempty for all t and thus $\mathcal{M}$ is *nicely* $\trianglerighteq$-rationalized by a continuous utility function u if $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$.

The following example illustrates the application of Theorem 3.

Example 4. Let $X = \mathbb{R}^n_+$ be the consumption space with n goods. To capture the notion that a consumer strictly prefers more goods to less, we let the product order $\geq$ to be underlying preorder. Then utility function u extends $\geq$ if and only if it is *strictly increasing*, in the sense that $u(x') > u(x)$ whenever $x' > x$. The order $\geq$ is continuous in the Euclidean topology on $\mathbb{R}^n_+$. Suppose that $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$, where A^t and B^t are compact sets; Theorem 3 guarantees that $\mathcal{M}$ can be nicely rationalized by a strictly increasing and continuous utility function if and only if it obeys the never-covered property under $\geq$.

There are other preorders besides the product order that could be natural in this setting. For example, $X = \mathbb{R}^n_+$ could be the space of contingent consumption, where the probabilities of each state are known (or part of the hypothesis). Then, based on those probabilities, different bundles in X could be ranked according to the first order stochastic dominance, i.e., $x \geq_{FSD} y$ if x first order stochastically dominates y. For example, suppose that the states are equiprobable; then $x \geq_{FSD} y$ and $y \geq_{FSD} x$ if the entries in y is a permutation of those in x. In this case, a utility

[18] Terminology: a preorder $\succsim$ is *continuous* (or *closed*) if $\{(x, y) \in X \times X : x \succsim y\}$ is a closed set in $X \times X$.

function that extends $\geq_{FSD}$ is simply a utility function that is strictly increasing in $\geq$ and symmetric.

Obviously, $\geq_{FSD}$ is a finer order than $\geq$ in the sense that $\geq \subset \geq_{FSD}$. It is also straightforward to check that $\geq_{FSD}$ is a continuous preorder. Suppose that $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ and A^t and B^t are compact for all t; by Theorem 3, $\mathcal{M}$ satisfies the never-covered property under $\geq_{FSD}$ if and only if it admits a nice rationalization by a continuous utility function that extends $\geq_{FSD}$.

Consider the example depicted in Figure 1(b), where $\hat{K}^p$ is weakly preferred to $\hat{K}^q$ in observation 1 and $\hat{K}^q$ is weakly preferred to $\hat{K}^r$ in observation 2. Since the never-covered property under $\geq$ is satisfied, these observations can be rationalized by a continuous and strictly increasing utility function. However, they are not rationalizable by a preference that extends $\geq_{FSD}$ when states 1 and 2 are equiprobable. Notice that for every $(a, b) \in \hat{K}^p$, there is $(a', b') \in \hat{K}^q$ such that either $(a', b') > (a, b)$ or $(b', a') > (a, b)$); thus $\hat{K}^p \subseteq \hat{K}^{q\downarrow\downarrow}$.[19] In formal terms, $\Phi(\{1\}) = \{1\}$, so the never-covered property is violated.

4 Application: Revealed price preference

One of the major themes in classical consumer theory is the recovery of the utility function from indirect utility. Formally, the question can be posed in the following way. Let $v : \mathbb{R}^n_{++} \to \mathbb{R}$ be a function. What necessary and sufficient conditions on v guarantee that

$$v(p) = \max\{u(x) : p \cdot x \leq 1\}$$

for some function $u : \mathbb{R}^n_+ \to \mathbb{R}$ (interpreted as the consumer's utility function)? This question has been thoroughly studied (see, for example, Krishna and Sonnenschein (1990) and Jackson (1986)) and it is well-known that the distinctive property that v necessarily satisfies is quasi-convexity.

Our objective in this section is to develop a finite analog of this question, in the sense that instead of recovering u from the function v we ask what conditions would allow us to recover a preference on the underlying bundles that are consistent with a *finite list* of preferences over prices. Of course, the quick and short answer to the issue before us is the never-covered property, but the additional structure of the consumer problem, with linear budget sets in Euclidean space, allows us to say more.

[19] From Figure 1(b), it is clear that $\hat{K}^p$ is contained in the interior of $\hat{K}^q \cup (\hat{K}^q)'$, where $(\hat{K}^q)'$ is the reflection of $\hat{K}^q$ on the 45 degree line.

We work with a data set with T observations, where at each observation t, the consumer reports either a weak or strong preference between two price vectors. Following our convention, if $t \in W$, then the consumer weakly prefers the price vector p^t to q^t. If $t \in S$, then the consumer strictly prefers the price vector p^t to q^t. Without loss of generality, we normalize the income of the consumer to be 1, so that the consumer's budget set at price $p \in \mathbb{R}^n_+$ is

$$L(p) := \left\{x \in \mathbb{R}^n_+ : p \cdot x \leq 1\right\}.$$

A preference for p^t over q^t means a preference for the budget $L(p^t)$ over $L(q^t)$. Thus the set of preferences over budget sets may be denoted by $\mathcal{M} = \left\{(L(p^t), L(q^t))\right\}_{t \in T}$. The next result is an application of Theorem 3 to this environment.

Theorem 4. *The following statements on $\mathcal{M} = \left\{(L(p^t), L(q^t))\right\}_{t \in T}$ are equivalent:*

1. *$\mathcal{M}$ can be rationalized by a locally nonsatiated preference on $\mathbb{R}^n_+$.*[20]
2. *$\mathcal{M}$ satisfies the never-covered property under the product order $\geq$.*
3. *$\mathcal{M}$ can be nicely rationalized by a strictly increasing, continuous, and concave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$.*

A straightforward application of Theorem 3 tells us that $\mathcal{M}$ satisfies the never-covered property under $\geq$ if and only if it can be rationalized by a strictly increasing and continuous utility function. The latter statement is replaced in Theorem 4 by both a weaker statement (rationalization by a locally nonsatiated preference) and a stronger statement (rationalization by a strictly increasing, continuous, and *concave* utility function). A proof of this result is in the Appendix, but it is worth noting the following here. In establishing that Statement (1) implies (2), we cannot simply appeal to the argument in Section 3.1 because in that case we make the assumption that the rationalizing preference $\succsim$ extends a preorder; however, the local nonsatiation assumption on $\succsim$ in this context allows us to retrace that argument, essentially because for any linear budget set L (assuming strictly positive prices), $L^{\downarrow} = L$ and $L^{\downarrow\downarrow}$ is the interior of L. As for the implication from (2) to (3), the linearity of the budget sets is crucial in guaranteeing that the rationalizing utility function can be chosen to be concave; our proof of that implication combines Theorem 3 with the Afriat's Theorem (see Afriat (1967)) which guarantees rationalization with a concave utility function.

[20] Terminology: A preference $\succsim$ is *locally nonsatiated* if for every $x \in \mathbb{R}^n_+$ and every open neighborhood N around x, there $x' \in N$ such that $x' \succ x$.

We know from standard consumer theory that quasiconvexity plays a crucial role in the characterization of indirect utility functions. The never-covered property could be thought of as the finite analog to quasiconvexity. In the case in which all the price preferences are strict, i.e., $T = S$, this connection is especially clear and is presented in Corollary 2 below.

To motivate the characterizing condition in Corollary 2, suppose that $\mathcal{M}$ is rationalized by a strictly increasing, continuous, and concave utility function u. This implies that the indirect utility function v is strictly decreasing in prices and quasiconvex. Given $T' \subseteq T$, suppose that $v(p^{t^*}) \geq v(p^t)$ for all $t \in T'$ and $v(q^{s^*}) \geq v(q^t)$ for all $t \in T'$. Since $T = S$, $v(p^{s^*}) > v(q^{s^*})$. Since v is quasiconvex, $v(q^{s^*}) \geq v(q)$ for all $q \in \text{conv}\Big(\{q^s\}_{s \in T'}\Big)$. Thus

$$v(p^{t^*}) \geq v(p^{s^*}) > v(q^{s^*}) \geq v(q)$$

for all $q \in \text{conv}\Big(\{q^s\}_{s \in T'}\Big)$. Since v is decreasing, we conclude that $p^{t^*} \ngeq q$ for any $q \in \text{conv}\Big(\{q^s\}_{s \in T'}\Big)$. It turns out that this quasiconvex-like property is precisely equivalent to the never-covered property.

Corollary 2. *The set of preferences over budget sets* $\mathcal{M} = \Big\{(L(p^t), L(q^t))\Big\}_{t \in T}$, *where* $T = S$, *is rationalizable by a strictly increasing, continuous, and concave utility function* $u : \mathbb{R}^n_+ \to \mathbb{R}$ *if and only if the following property holds: for any nonempty* $T' \subseteq T$,

there exists some $t \in T'$ *such that* $p^t \ngeq q$ *for any* $q \in \text{conv}\Big(\{q^s\}_{s \in T'}\Big)$.

The proof of this result, as well as its extension to the case in which W is nonempty, can be found in the Appendix.

5 Application: Coarse rationalizability

So far in this paper, we have considered the rationalization of a set of menu preference pairs. In this section, we discuss a formally related but economically distinct issue, namely, *the rationalization of choices from menus*. Our contribution is to provide a method for testing rationalizability in situations where observations are *coarse*, in a sense we shall make specific. Among other things, we provide an extension of the Afriat's Theorem to this environment.

5.1 Four concepts of rationalization

Suppose that at observation t, there is a menu C^t and a set $A^t \subseteq C^t$. Two notions of rationalization are commonly used in analyses of this type. The **first concept** requires a preference $\succsim$ such that $A^t = \max(C^t; \succsim)$ for all $t \in T$; the Richter's Theorem (see Richter (1966)) characterizes data sets which are rationalizable in this sense. The **second concept** requires a preference $\succsim$ such that $A^t \subseteq \max(C^t; \succsim)$ for all $t \in T$; the Afriat's Theorem (and its generalizations to nonlinear domains) characterize data sets that satisfy this concept of rationalization. Loosely speaking, the first notion of rationalization is the one most commonly used in the theoretical revealed preference literature; on the other hand, empirical work using revealed preference has mostly relied on the second, weaker notion, which is unsurprising since it does not posit that the observer has observed all the optimal choices, but only one (or some) of them.

A **third concept** of rationalization has been characterized in Fishburn (1976), where the set of optimal points is required to be contained in A^t; in other words, $\max(C^t; \succsim) \subseteq A^t$. Obviously, Fishburn's concept generalizes the one in Richter's Theorem by allowing some elements of A^t to be nonoptimal, but it retains the requirement that nothing outside of A^t is optimal. This suggests that a **fourth concept** of rationalization may be useful in empirical applications: one that allows for the possibility that some elements in A^t are nonoptimal (following Fishburn) and also that some elements outside of A^t are optimal (following Afriat). In formal terms, it requires that $\max(C^t; \succsim) \cap A^t \neq \emptyset$.

The revealed preference literature since the 1970s have by and large neglected Fishburn's rationalization concept and its corresponding result. We think that Fishburn's concept, as well as the relaxation of that concept which we just proposed, deserves notice because they are relevant to empirical applications of revealed preference.

These concepts are applicable whenever there are *coarse observations*, where the observer knows (or hypothesizes) that there is an optimal choice found in A^t, but is agnostic about precisely which alternatives within A^t are optimal. There are at least three broad scenarios where it is useful to think of coarse observations.

(1) The most obvious cases are those where the observations are simply known to be imprecise. For example, a researcher may have information on how much is spent on broad categories of goods, without knowing the allocation within each category. Alternatively, a researcher may have records on a consumer's credit card purchases, which puts a *lower* bound on how money is spent each month on different goods, but

does not provide the precise breakdown of monthly expenditure since there could be goods bought with cash.

(2) There could be situations where some alternative y^t is recorded as the choice from C^t but, in testing for rationality or estimating the preference, the researcher may wish to accommodate the possibility that choices were observed with error; this could be accomplished by defining a neighborhood A^t around y^t (in some sense appropriate to the specific context) and then checking if there is a preference $\succsim$ with $\max(C^t; \succsim) \cap A^t \neq \emptyset$ for all $t \in T$.

(3) In experimental settings, it is common to find subjects whose choice behavior are not exactly consistent with rationality. Since the choices y^t are typically observed perfectly, it is implausible to attribute the rationality violations to observational errors. However, one could still use the size of the neighborhood A^t (suitably measured) as a way of comparing the rationality of different experimental subjects; those who require large A^ts to rationalize their behavior can be deemed less rational than those where A^t is just a small neighborhood of y^t.

5.2 Coarse data sets and menu preferences

We consider an observer who has a finite set of coarse observations of a decision maker's choices. We denote a *coarse data set* by $\mathcal{O} = \left\{(A^t, C^t)\right\}_{t\in T}$ where $\{W, S\}$ is a partition of T and for each $t \in T$ we have $\emptyset \neq A^t \subseteq C^t$. The interpretation is as follows. When $t \in W$, A^t contains at least one choice of the decision maker in C^t; when $t \in S$, A^t contains all the choices of the decision maker in C^t. The observer would like to recover a preference $\succsim$ that rationalizes the data in the following sense.

Definition 7. *A preference $\succsim$ on X rationalizes the coarse data set $\mathcal{O} = \left\{(A^t, C^t)\right\}_{t\in T}$ if*

1. $\max(C^t; \succsim) \cap A^t \neq \emptyset$ *for each* $t \in W$, *and*
2. $\max(C^t; \succsim) \subseteq A^t$ *for each* $t \in S$.

If $\succsim$ exists, we say that $\mathcal{O}$ is rationalizable. *If $\succsim$ can be chosen to rationalize $\mathcal{O}$ and extend a given preorder $\trianglerighteq$, then $\mathcal{O}$ is $\trianglerighteq$-rationalizable.*

Obviously, conditions (1) and (2) in this definition correspond precisely to the fourth and third concepts of rationalization discussed in Section 5.1. Note that if $T = W$, then every coarse data set is trivially rationalized by a preference that is indifferent across all alternatives. So in the case, the rationalizability problem is interesting only if the preference is required to be locally nonsatiated or to extend some preorder $\trianglerighteq$.

Checking if a coarse data set is rationalizable is straightforward, given the results on menu preferences we have developed in Section 3. Indeed, for any coarse data set $\mathcal{O} = \left\{(A^t, C^t)\right\}_{t\in T}$, we could construct the following set $\mathcal{M}^*$ of menu preference pairs: the menu A^t is weakly preferred to the menu C^t for all $t \in T$ and A^t is strictly preferred to the menu $C^t \setminus A^t$ for $t \in S$. Clearly, $\mathcal{M}^*$ is rationalized by a preference $\succsim$ if and only if the coarse data set $\mathcal{O}$ is rationalizable by the same preference $\succsim$. Thus every result we have on the rationalizability (or $\trianglerighteq$-rationalizability) of the set of menu preference pairs $\mathcal{M}^*$ has an analog for $\mathcal{O}$.

In the following subsection, we extend the Afriat's Theorem to coarse data sets.

5.3 A generalization of Afriat's theorem

We consider a data set $\mathcal{O} = \left\{(A^t, L(p^t, y^t))\right\}_{t\in T}$ where for each $t \in T$, $p^t \in \mathbb{R}^n_{++}$ is the price vector, y^t is the total expenditure, and $L(p^t, y^t) := \{x \in \mathbb{R}^n_+ : p^t \cdot x \leq y^t\}$ is the budget set at observation t.[21] Departing from the standard setting of Afriat's Theorem, the observer does not know the exact choice of the consumer and only knows that the choice lies in $A^t \subseteq L(p^t)$. The following result provides us with a test of coarse rationalizability in this setting.

Theorem 5. *Let $\mathcal{O} = \left\{(A^t, L(p^t, y^t))\right\}_{t\in T}$ be a coarse data set where $T = W$ and $A^t \subseteq L(p^t, y^t)$ for all $t \in T$. The following statements are equivalent:*

1. *$\mathcal{O}$ can be rationalized by a locally nonsatiated preference.*
2. *$\mathcal{O}$ satisfies the never-covered property under the product order $\geq$.*[22]
3. *$\mathcal{O}$ can be rationalized by a strictly increasing, continuous, and concave utility function.*

Example 5. In studies of consumer demand, a researcher would often not have information on the demand for every relevant good. A common way to address this issue is to perform some aggregation procedure across goods, even though this approach is strictly valid only under stringent conditions on the utility function and/or the pattern of prices changes.

To be more specific, suppose that at observation t, the information available consists of the prices of all goods $p^t \in \mathbb{R}^n_{++}$, the demand for the first $m-1$ goods, and the total expenditure on the remaining goods (which we denote by $c^t_{m,n}$). In

[21] Note that we departing from the convention and notation of the previous section by *not* normalizing expenditure at 1. This presentation is more appropriate in this section to highlight the fact that total expenditure y^t is part of the observer's data.

[22] In this statement, we are interpreting $\mathcal{O}$ as a set of weak menu preferences.

other words, the actual demand for goods m, $m+1, \ldots, n$ is not observed. To get round this problem, the researcher could construct a price index for those goods, $\bar{p}^t_m$, which would be a function of their prices $(p^t_m, p^t_{m+1}, \ldots, p^t_n)$, with the corresponding demand for the composite good being $\bar{x}^t_m = c^t_{m,n}/\bar{p}^t_m$. In this way, the researcher creates a data set of the standard form, with prices $(p^t_1, p^t_2, \ldots, \bar{p}^t_m)$ and demand $(x^t_1, x^t_2, \ldots, \bar{x}^t_m)$ for m goods at each observation.

Coarse data sets offer a potentially useful alternative approach to tackle this problem. At observation t, the researcher observes x^t_i for $i = 1, \ldots, m-1$ and $c^t_{m,n}$. Thus the demand of the consumer must lie in the set

$$A^t = \{x \in \mathbb{R}^n_+ : x_i = x^t_i \text{ for } i = 1, \ldots, m-1 \text{ and } \textstyle\sum_{i=m}^n p_i x_i = c^t_{m,n}\}.$$

The corresponding coarse data set is $\mathcal{O} = \{(A^t, L(p^t, y^t))\}_{t \in T}$, where $y^t = \sum_{i=1}^{m-1} p^t_i x^t_i + c^t_{m,n}$. This can be analyzed using Theorem 5.

As an illustration, suppose that $\mathcal{O}$ consists of two observations where

$$\begin{array}{llll} p^1 = (2, 2.5, 3.5), & x^1_1 = 1.5, & c^1_{2,3} = 9, & y^1 = 12 \\ p^2 = (4, 3, 3), & x^1_1 = 3, & c^1_{2,3} = 4.5, & y^2 = 16.5. \end{array}$$

This data set is coarse rationalizable. Indeed, the bundle $\tilde{x} = (1.5, 9/2.5, 0)$ is in A^1 but $p^2 \cdot \tilde{x} = 16.8 > 16.5$, so it is not in $L(p^2, y^2)$. Given that there are just two observations, this is enough to guarantee that $\mathcal{O}$ satisfies the never-covered property under $\geq$.

On the other hand, suppose we were to aggregate goods 2 and 3 into a composite commodity, with the price of the composite good fixed at 3 at both observations 1 and 2. Then the demand for the composite good at these observations are $\bar{x}^1_2 = 9/3 = 3$ and $\bar{x}^2_2 = 4.5/3 = 1.5$. The corresponding two-good data set has

$$\begin{array}{lll} p^1 = (2, 3), & x^1 = (1.5, 3), & y^1 = 12; \\ p^2 = (4, 3), & x^2 = (3, 1.5) & y^2 = 16.5. \end{array}$$

It is straightforward to check that this data set violates GARP and is not rationalizable.

Example 6. Consider a researcher who observes that the consumer chooses the bundle x^t from the budget set $L(p^t, y^t)$. To allow for the possibility that x^t was observed with error, the researcher could allow for the true consumption bundle to

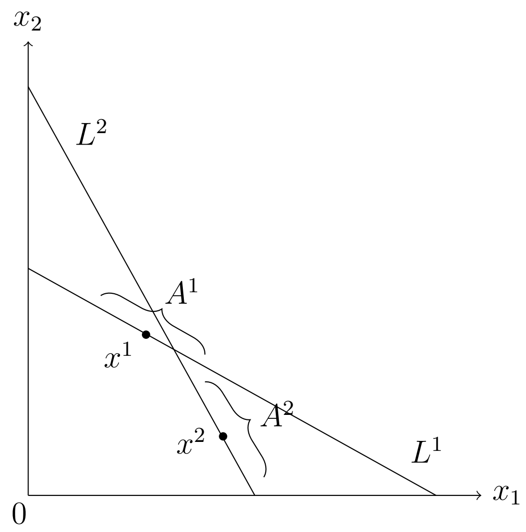

Figure 2: The data set $\{(x^1, L^1), (x^2, L^2)\}$ is not rationalizable, but $\{(A^1, L^1), (A^2, L^2)\}$ (as depicted) is rationalizable.

be in the set

$$A^t = \{x \in L(p^t, y^t) : \ p^t \cdot x = y^t \text{ and } |p_i^t x_i - p_i^t x_i^t| \leq k y^t \text{ for all } i\},$$

where $k \in [0, 1]$. In other words, the expenditure on good i is allowed to deviate from $p_i^t x_i^t$ but not by more than ky^t. This is illustrated in Figure 2, where the 'original' data set $\{(x^1, L^1), (x^2, L^2)\}$ is not rationalizable, but $\{(A^1, L^1), (A^2, L^2)\}$ (as depicted) is rationalizable. More generally, our extension of the Afriat's Theorem provides a way to check if $\mathcal{O} = \left\{(A^t, L(p^t, y^t))\right\}_{t \in T}$ is rationalizable.

6 Application: Multiple preferences

In this section, we investigate the observable restrictions of the multiple preferences model; see, for example, Aizerman and Malishevski (1981) and Salant and Rubinstein (2008). In contrast with the single preference model, the choice behavior of the DM may be a result of multiple rationales. Formally, the DM has a set Π of strict preferences, and she chooses

$$f_\Pi(A) := \Big\{x : x = \max(A; \succ) \text{ for some } \succ \in \Pi\Big\}$$

in each menu A.

We represent the observed choice behavior of the DM by a data set (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(A)$ is the collection of alternatives that the DM chooses in $A \in \Sigma$. We say that (Σ, f) is rationalizable by multiple preferences if there exists a set Π of strict preferences such that

$$f_\Pi(A) = f(A)$$

for all $A \in \Sigma$. For notational simplicity, we denote by $g(A) := A \setminus f(A)$ the collection of alternatives that are not chosen in $A \in \Sigma$. In what follows, we identify necessary and sufficient conditions under which (Σ, f) is rationalizable by multiple preferences by applying our results in Section 3.

For a given data set (Σ, f), let us first consider the following data set of menu preferences

$$\mathcal{O} := \Big\{(f(A), g(A))\Big\}_{A \in \Sigma},$$

where all menu preferences are strict. It is easy to see that if (Σ, f) is rationalizable by multiple preferences, then there necessarily exists a strict preference $\succ$ such that $\max(A; \succ) \in f(A)$ for all $A \in \Sigma$. In other words, $\mathcal{O}$ must be rationalizable by a strict preference. However, this is not sufficient for rationalizability by multiple preferences, as illustrated by the following example.

Example 7 (A data set that is not rationalizable by multiple preferences)**.** Let $X = \{x, y, z\}$. Consider the data set (Σ, f) consisting of the following three observations:[23]

(1) $f(\{x, y\}) = x$;

(2) $f(\{y, z\}) = y$;

(3) $f(\{x, y, z\}) = \{x, z\}$.

Consider the data set of menu preferences $\mathcal{O} = \Big\{(x, y), (y, z), (\{x, z\}, y)\Big\}$. It follows from Corollary 1 that $\mathcal{O}$ is rationalizable by a strict preference (for example, by the strict preference $x \succ y \succ z$). However, (Σ, f) is not rationalizable by multiple preferences. To see this, suppose to the contrary that (Σ, f) is rationalizable by multiple preferences. By definition, there exists a set Π of strict preferences such that $f_\Pi(A) = f(A)$ for all $A \in \Sigma$. The first observation $f(\{x, y\}) = x$ reveals that $x \succ y$ for all $\succ \in \Pi$, and the second observation $f(\{y, z\}) = y$ reveals that $y \succ z$ for all $\succ \in \Pi$. By transitivity, it must be that $x \succ z$ for all $\succ \in \Pi$, which contradicts with the third observation that $f(\{x, y, z\}) = \{x, z\}$.

We now present a characterization of the multiple preferences model as follows. By Corollary 1, the data set of menu preferences $\mathcal{O} = \Big\{(f(A), g(A))\Big\}_{A \in \Sigma}$ is rationalizable by a strict preference if and only if for all $\emptyset \neq \Sigma' \subseteq \Sigma$,

$$\cup_{A \in \Sigma'} f(A) \not\subseteq \cup_{A \in \Sigma'} g(A).$$

[23] We abuse the notation by suppressing the set delimiters, e.g., writing x rather than $\{x\}$.

As illustrated by Example 7, this is not sufficient. We strengthen this condition as follows: for any nonempty $\Sigma' \subseteq \Sigma$ and $B \in \Sigma$,

$$\Big(\cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A) \Big) \subseteq B \Rightarrow f(B) \cap \Big(\cup_{A\in\Sigma'} g(A) \Big) = \emptyset.$$ [24]

To understand the condition, note that it is plainly a necessary condition. Indeed, suppose that the data set is rationalizable by multiple preferences. If an alternative is not chosen in A, then under each preference, it is dominated by some alternative chosen in A. Hence, any alternative in $g(A)$ is not chosen in a menu that contains $f(A)$. This implies that the set of choices made in $\cup_{A\in\Sigma'} A$ does not contain any alternative in $\cup_{A\in\Sigma'} g(A)$. Furthermore, since $f(A)$ contains all maximal alternatives under each preference in A, choices made in $\cup_{A\in\Sigma'} A$ should be a subset of $\cup_{A\in\Sigma'} f(A)$, and thus a subset of $\cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A)$. By a similar argument, any alternative in $\cup_{A\in\Sigma'} g(A)$ is not chosen in any menu containing $\cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A)$, as stated by the condition. Theorem 6 shows that this new condition is both necessary and sufficient for the rationalizability of a data set by multiple preferences.

From a practical perspective, we provide a divide-and-conquer approach to test whether a give data set (Σ, f) is rationalizable by multiple preferences. Rationalizability by multiple preferences requires that for each $A \in \Sigma$ and $x \in f(A)$, there exist a strict preference such that x is the maximal element in A according to this strict preference. Furthermore, the maximal element in any other set $A' \in \Sigma$ according to this strict preference must lie in $f(A')$. Thus, for each $A \in \Sigma$ and $x \in f(A)$, the following data set of menu preferences

$$\mathcal{O}_{A,x} := (x, A \setminus x) \cup \Big\{ (f(A'), A' \setminus f(A')) \Big\}_{A'\in\Sigma,\, A'\neq A},$$

where all menu preferences are strict, must be rationalizable by a strict preference. Let

$$\mathfrak{D} := \{\mathcal{O}_{A,x}\}_{A\in\Sigma,\, x\in f(A)}.$$

A necessary condition for the data set (Σ, f) to be rationalizable by multiple preferences is that each $\mathcal{O}_{A,x}$ in $\mathfrak{D}$ is rationalizable by a strict preference. Theorem 6 below shows that the converse is also true. Recall that it is straightforward to check whether a data set of menu preferences is rationalizable by a strict preference

[24] To see that this is a strengthening of the previous condition, suppose that there exists some nonempty $\Sigma' \subseteq \Sigma$ such that $\cup_{A\in\Sigma'} f(A) \subseteq \cup_{A\in\Sigma'} g(A)$. Fix an arbitrary nonempty set $B \in \Sigma'$. Since $\cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A) = \emptyset \subseteq B$, it follows from the new condition that $f(B) \cap \big(\cup_{A\in\Sigma'} g(A) \big) = \emptyset$. We have arrived at a contradiction, since $f(B) \subseteq \cup_{A\in\Sigma'} f(A) \subseteq \cup_{A\in\Sigma'} g(A)$.

(see Section 3.3). Thus, it is straightforward to check whether a data set (Σ, f) is rationalizable by multiple preferences using the divide-and-conquer approach.

Theorem 6. *The following statements are equivalent:*

(1) (Σ, f) *is rationalizable by multiple preferences.*

(2) For any nonempty $\Sigma' \subseteq \Sigma$ *and* $B \in \Sigma$,

$$\Big(\cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A) \Big) \subseteq B \Rightarrow f(B) \cap \Big(\cup_{A\in\Sigma'} g(A) \Big) = \emptyset.$$

(3) Each $\mathcal{O}_{A,x}$ *in* $\mathfrak{D}$ *is rationalizable by a strict preference.*

Theorem 6 holds for any data set (Σ, f). In the special case of complete data, that is, when $\Sigma = \mathcal{X}$, Aizerman and Malishevski (1981) show that the multiple preferences model could be characterized by the following two axioms:

Chernoff: $A \subseteq B \Rightarrow f(B) \cap A \subseteq f(A)$ for all $A, B \in \mathcal{X}$.

Aizerman: $f(B) \subseteq A \subseteq B \Rightarrow f(A) \subseteq f(B)$ for all $A, B \in \mathcal{X}$.

In words, the Chernoff axiom says that a best choice in some set is still best if the set shrinks. The Aizerman axiom says that deleting from a given set some choices outside the choice set cannot make new choices chosen. We establish this result as a corollary of Theorem 6 by showing that our characterization for any data set (complete or incomplete) follows from the Chernoff axiom and the Aizerman axiom in the case of complete data.

Corollary 3 (Aizerman and Malishevski (1981))**.** *The data set* $(\mathcal{X}, f)$ *is rationalizable by multiple preferences if and only if it satisfies the Chernoff axiom and the Aizerman axiom.*

7 Application: Minimax regret

The decision criterion of minimax regret is first suggested in Savage (1951)'s reading of Wald (1950) to model a DM who anticipates regret and thus incorporates in her choice the desire to minimize the worst-case regret. In this section, we investigate the observable restrictions of the minimax regret model by applying our results in Section 3.

We adopt the standard framework of the state space that has been discussed by, for example, Kreps (1979) and Dekel et al. (2001). We consider any nonempty finite set X. Let U denote a state space with a typical element u. To economize on notation, we also write u to denote the utility function of the DM associated with the state u. The regret of choosing x relative to y is $u(y) - u(x)$ if the state is u, the worst-case regret of choosing x relative to y is

$$\phi_U(x, y) := \max_{u \in U} \Big\{u(y) - u(x)\Big\},$$

and the worst-case regret of choosing x in a menu A is

$$\max_{y \in A} \phi_U(x, y).$$

The DM who uses the minimax regret decision criterion chooses all alternatives that generate the lowest worst-case regret. That is, the DM chooses

$$f_U(A) = \arg\min_{x \in A} \Big\{\max_{y \in A} \phi_U(x, y)\Big\}$$

in each menu A.[25]

We represent the observed choice behavior of the DM by (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(A)$ is the choice of the DM in $A \in \Sigma$. For the sake of simplicity, we assume that f is a choice function.[26] We say that (Σ, f) is rationalizable under the minimax regret model if there exists a finite set U of utility functions such that

$$f(A) = \arg\min_{x \in A} \Big\{\max_{y \in A} \phi_U(x, y)\Big\} \tag{3}$$

for all $A \in \Sigma$.

Given the flexibility to construct the set of utility functions, the readers might wonder, whether there are any observable restrictions of the minimax regret model. To get this out of the way, let us first present a data set (Σ, f) that is not rationalizable under the minimax regret model. We shall also use this data set in Example 8 (Continued) to illustrate the approach that we establish in this section.

Example 8 (A data set that is not rationalizable under the minimax regret model)**.** Let $X = \{x, y, z, w\}$. Consider a data set (Σ, f) consisting of the following three observations: $f(X) = x$, $f(X \setminus z) = y$, and $f(X \setminus w) = y$.

[25] Our minimax regret model differs from that in Milnor (1954) and Stoye (2011), who axiomatize the preference ordering represented by (the negative of)

$$\max_{s \in \mathcal{S}} \Big\{\max_{g \in M} u \circ g(s) - u \circ f(s)\Big\},$$

where f and g are acts in a menu M, $\mathcal{S}$ is a state space, and u is an expected utility functional.

[26] The case of a choice correspondence can be accommodated using similar logic.

Suppose that (Σ, f) is rationalizable under the minimax regret model, say, by the set U of utility functions. In what follows, we show that (1) $f(X) = x$ and $f(X \setminus z) = y$ imply that $\phi_U(y, z) > \phi_U(y, w)$; and (2) $f(X) = x$ and $f(X \setminus w) = y$ imply that $\phi_U(y, w) > \phi_U(y, z)$. Thus, we have a contradiction. Since $f(X) = x$, x generates a lower worst-case regret in X than y does. It follows that the worst-case regret of x in $X \setminus z$ is lower than the worst-case regret of y in X. Since $f(X \setminus z) = y$, the worst-case regret of y in $X \setminus z$ is lower than the worst-case regret of x in $X \setminus z$. Therefore, the worst-case regret of y in X and $X \setminus z$ are not the same. This is possible only if the worst-case regret of y relative to z is higher than the worst-case regret of y relative to w. Similarly, since $f(X) = x$ and $f(X \setminus w) = y$, we can conclude that the worst-case regret of y relative to w is higher than the worst-case regret of y relative to z.

We present a lemma that simplifies our analysis below. This lemma shows that, without loss of generality, we can work with the following choice function:

$$f(A) = \arg\min_{x \in A} \left\{ \max_{y \in A \setminus x} \phi_U(x, y) \right\} \tag{4}$$

for all $A \in \Sigma$. The proof is trivial and purely algebraic, and hence omitted.

Lemma 1. *There exists a finite set U of utility function such that (3) holds for all $A \in \Sigma$ if and only if (4) holds for all $A \in \Sigma$ under the same U.*

By Lemma 1, the rationalizability of a data set (Σ, f) under the minimax regret model is equivalent to the existence of a finite set U of utility functions such that

$$\max_{y \in A \setminus f(A)} \phi_U(f(A), y) < \max_{y \in A \setminus x} \phi_U(x, y)$$

for all $A \in \Sigma$ and $x \in A \setminus f(A)$. For notational simplicity, throughout the rest of this section, we shall write (x, y) rather than $\phi_U(x, y)$ to denote the worst-case regret of x relative to y. This should not cause any confusion.

Interestingly, the rationalizability of a data set under the minimax regret model is related to menu preferences. For any (Σ, f), we construct its corresponding data set $\bar{\mathcal{O}}$ as follows: for each $A \in \Sigma$ and $x \in A$ with $x \neq f(A)$, let $(A_i, B_i) \in \bar{\mathcal{O}}$ where

$$\bar{A}_i = x \times (A \setminus x) \text{ and } \bar{B}_i = f(A) \times (A \setminus f(A)).$$

The interpretation of the data point $(\bar{A}_i, \bar{B}_i)$ is that the menu $\bar{A}_i$ is strictly better than $\bar{B}_i$ in the numeric order, which captures that alternative x which is not chosen in A generates a higher worst-case regret in A than $f(A)$ does. Theorem 7 below makes the formal link between the rationalizability of a data set under the minimax regret model and the rationalizability of menu preferences.

Theorem 7. *The data set* (Σ, f) *is rationalizable under the minimax regret model if and only if the corresponding data set* $\bar{\mathcal{O}}$ *is rationalizable by a strict preference.*

Recall that it is straightforward to check whether a data set of menu preferences is rationalizable by a strict preference (see Section 3.3). Thus, it is straightforward to check whether a data set (Σ, f) is rationalizable under the minimax regret model. We now revisit Example 8 to illustrate how to use Theorem 7 to show that the data set is not rationalizable under the minimax regret model.

Example 8 (Continued)**.** We construct the corresponding data set $\bar{\mathcal{O}}$ as follows:

$$
\begin{aligned}
\bar{A}_1 &= \Big\{(y,x),(y,z),(y,w)\Big\}, & \bar{B}_1 &= \Big\{(x,y),(x,z),(x,w)\Big\};\\
\bar{A}_2 &= \Big\{(z,x),(z,y),(z,w)\Big\}, & \bar{B}_2 &= \Big\{(x,y),(x,z),(x,w)\Big\};\\
\bar{A}_3 &= \Big\{(w,x),(w,y),(w,z)\Big\}, & \bar{B}_3 &= \Big\{(x,y),(x,z),(x,w)\Big\};\\
\bar{A}_4 &= \Big\{(x,y),(x,w)\Big\}, & \bar{B}_4 &= \Big\{(y,x),(y,w)\Big\};\\
\bar{A}_5 &= \Big\{(w,x),(w,y)\Big\}, & \bar{B}_5 &= \Big\{(y,x),(y,w)\Big\};\\
\bar{A}_6 &= \Big\{(x,y),(x,z)\Big\}, & \bar{B}_6 &= \Big\{(y,x),(y,z)\Big\};\\
\bar{A}_7 &= \Big\{(z,x),(z,y)\Big\}, & \bar{B}_7 &= \Big\{(y,x),(y,z)\Big\}.
\end{aligned}
$$

By Corollary 1, $\bar{\mathcal{O}}$ is not rationalizable by a strict preference. In particular, $\left(\bar{A}_1 \cup \bar{A}_4 \cup \bar{A}_6\right) \subseteq \left(\bar{B}_1 \cup \bar{B}_4 \cup \bar{B}_6\right)$. By Theorem 7, (Σ, f) is not rationalizable under the minimax regret model.

A Appendix

Proof of Theorem 1. We have argued in the main text that Statement 1 implies Statement 2. In what follows, we show that Statement 2 implies Statement 3 and that Statement 3 implies Statement 1.

(Statement 2 $\Rightarrow$ Statement 3) Suppose that $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$. We shall explicitly provide a way of selecting x^t in A^t for each $t \in T$ such that $\{x^t\}_{t\in T}$ is a no-cycling selection.

For ease of notation, let us denote by $\mathcal{E}(T')$ the set of alternatives that are revealed to be dominated through the procedure of iterated exclusion of dominated observations, i.e.,

$$\mathcal{E}(T') := B\left(T'\right)^{\downarrow\downarrow} \bigcup B\left(\left(T' \cap S\right) \cup \Phi(T')\right)^{\downarrow}.$$

Since $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, for any nonempty $T' \subseteq T$, $\Phi(T')$ is strict subset of T', which implies that $A\left(T'\right) \setminus \mathcal{E}(T') \neq \emptyset$.

Let $T_1 := T$ and $S_1 := A(T_1) \setminus \mathcal{E}(T_1)$. We proceed by induction. Suppose that we have constructed T_k and S_k for some $k \geq 1$. If $T_k \neq \emptyset$, we construct T_{k+1} and S_{k+1} as follows:

$$\begin{aligned} &T_{k+1} := \Phi(T_k) = \{t \in T_k : A^t \subseteq \mathcal{E}(T_k)\}, \text{ and} \\ &S_{k+1} := A(T_{k+1}) \setminus \mathcal{E}(T_{k+1}). \end{aligned}$$

Since $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, if $T_k \neq \emptyset$, then $T_{k+1} = \Phi(T_k)$ is a strict subset of T_k and $S_k = A(T_k) \setminus \mathcal{E}(T_k) \neq \emptyset$. The construction stops when $T_{k^*} \neq \emptyset$ and $T_{k^*+1} = \emptyset$ for some k^*.

We are now ready to select x^t in A^t for each $t \in T$ such that $\{x^t\}_{t\in T}$ is a no-cycling selection. For each $1 \leq k \leq k^*$, let $V_k := T_k \setminus T_{k+1}$ denote the collection of observations that are eliminated when constructing T_{k+1} from T_k. Clearly, $\{V_k\}_{k=1}^{k^*}$ is a partition of T. By definition, for each k and each $t \in V_k = T_k \setminus T_{k+1}$, we have $A^t \setminus \mathcal{E}(T_k) \neq \emptyset$ and hence $A^t \cap S_k = A^t \cap (A(T_k) \setminus \mathcal{E}(T_k)) \neq \emptyset$.

For each k and each $t \in V_k = T_k \setminus T_{k+1}$, we pick an arbitrary $x^t \in A^t \cap S_k$. We proceed to verify that the revealed preference relations defined on $\{x^t\}_{t\in T}$ obey the no-cycling property. Let $k(t)$ be the corresponding index k such that $t \in V_k$. It suffices to show that (1) $x^t R x^{t'}$ implies that $k(t) \leq k(t')$; and (2) $x^t P x^{t'}$ implies that $k(t) < k(t')$. Suppose that $x^t R x^{t'}$ but $k(t) > k(t')$. Then $t \in \Phi(T_{k(t')})$ due to the construction of $\{V_k\}_{k=1}^{k^*}$. It follows that $A^t \subseteq \mathcal{E}(T_{k(t')})$ and $B^{t\downarrow} \subseteq \mathcal{E}(T_{k(t')})$. Since $x^t R x^{t'}$, we have $x^{t'} \in B^{t\downarrow} \subseteq \mathcal{E}(T_{k(t')})$, which contradicts with $x^{t'} \in S_{k(t')} = A(T_{k(t')}) \setminus \mathcal{E}(T_{k(t')})$. Hence, $x^t R x^{t'}$ implies $k(t) \leq k(t')$. Suppose that $x^t P x^{t'}$ but $k(t) \geq k(t')$. If $k(t) > k(t')$, then we have the same contradiction as argue above. If $k(t) = k(t') = k$ for some k, then both x^t and $x^{t'}$ belong to S_k. Since $S_k = A(T_k) \setminus \mathcal{E}(T_k)$ and $B(T_k)^{\downarrow\downarrow} \cup B(T_k \cap S)^{\downarrow} \subseteq \mathcal{E}(T_k)$, we have

$$x^t, x^{t'} \in S_k \subseteq A(T_k) \setminus \Big(B(T_k)^{\downarrow\downarrow} \cup B(T_k \cap S)^{\downarrow}\Big).$$

But this is impossible since $x^t P x^{t'}$ implies that either $x^{t'} \in B^{t\downarrow\downarrow}$ or $t \in S$ and $x^{t'} \in B^{t\downarrow}$. Hence, $x^t P x^{t'}$ implies $k(t) < k(t')$.

(Statement 3 $\Rightarrow$ Statement 1) Suppose that $\mathcal{M}$ admits a no-cycling selection $\{x^t\}_{t\in T}$. Let R^* be the binary relation on X where $\hat{x}\, R^*\, \hat{y}$ if there is $t \in T$ such that $\hat{x} = x^t$ and $\hat{y} \in B^t$. Let $\succsim^*$ be the transitive closure of the binary relation $R^* \cup \trianglerighteq$. By the Szpilrajn's extension theorem (see Szpilrajn (1930)), $\succsim^*$ admits an extension $\succsim$.[27]

We claim that the preference $\succsim$ has two properties: (1) it rationalizes the data set and (2) it extends $\trianglerighteq$. It follows from the construction that $x^t \succsim B^t$ for all $t \in T$, so it remains to show that $x^t \succ B^t$ for $t \in S$. To show (1), suppose to the contrary that for some $t' \in S$, $x^{t'} \sim y$ for some $y \in B^{t'}$. Given that $\succsim$ extends $\succsim^*$, this can only occur if $y \succsim^* x^{t'}$. This means there is $t'' \in T$ such that $y \trianglerighteq x^{t''} \succsim^* x^{t'}$. Therefore we obtain $x^{t'}\, P\, x^{t''} \succsim^* x^{t'}$, which is excluded by the no-cycling property. This completes the proof of (1). To show (2), note that $x \succsim y$ if $x \trianglerighteq y$ by construction, so it remains to show that $x \succ y$ if $x \triangleright y$. Suppose instead we have $x \triangleright y$ but $x \sim y$. This can only occur if $y \succsim^* x$. Since $\trianglerighteq$ is a preorder, if $x \triangleright y$ and $y \succsim^* x$, there must be $t', t'' \in T$ such that $y \trianglerighteq x^{t''} \succsim^* x^{t'} \trianglerighteq x$. So we obtain $x^{t'} \trianglerighteq x \triangleright y \trianglerighteq x^{t''}$, which means that $x^{t'}\, P\, x^{t''}$, which is incompatible with $x^{t''} \succsim x^{t'}$, given the no-cycling property. $\square$

Proof of Proposition 1. We prove the first statement below. The second statement is an immediate implication of the first statement. Fix a nonempty $T' \subseteq T$ and $\Phi \subsetneq T'$ such that for any $t \in T' \setminus \Phi$,

$$A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi)^{\downarrow}.$$

To show that $\Phi(T') \subseteq \Phi$, we proceed by induction. Obviously,

$$\Phi^0(T') = \emptyset \subseteq \Phi,$$
$$\Phi^1(T') = \left\{t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}\right\} \subseteq \Phi.$$

Suppose that we have $\Phi^m(T') \subseteq \Phi$ for some m. It follows that

$$B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^m(T'))^{\downarrow} \subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi)^{\downarrow}.$$

Thus, $t \in T' \setminus \Phi$ implies $t \in T' \setminus \Phi^{m+1}(T')$, which further implies that $\Phi^{m+1}(T') \subseteq \Phi$. By induction, we have $\Phi(T') \subseteq \Phi$. $\square$

Proof of Proposition 2. By Theorem 1, $\mathcal{M}$ is $\trianglerighteq$-rationalizable if and only if it satisfies the never-covered property under $\trianglerighteq$. Thus, it suffices to show that $\mathcal{M}$

[27] This means that there is a complete preorder $\succsim$ such that $x \succsim y$ if $x \succsim^* y$ and $x \succ y$ if $x \succ^* y$ where $\succ$ is the asymmetric part of $\succsim$ and $\succ^*$ the asymmetric part of $\succsim^*$.

satisfies the never-covered property under $\trianglerighteq$ if and only if the ALGORITHM outputs $\trianglerighteq$-*Rationalizable*.

(The only if-part) If $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, then $\Phi(T') \neq T'$ for any nonempty $T' \subseteq T$. Thus, Case (c) never occurs when we run the ALGORITHM on this data set. Furthermore, T^k is strictly decreasing in k in the set inclusion sense, and $T^{k^*} = \emptyset$ for some k^*. Therefore, the ALGORITHM outputs $\trianglerighteq$-*Rationalizable*.

(The if-part) Conversely, suppose that the ALGORITHM outputs $\trianglerighteq$-*Rationalizable*. We then have a sequence of subsets of T, $\{T^0, T^1, \ldots, T^{k^*}\}$, where $T^k = \Phi(T^{k-1}) \subsetneq T^{k-1}$ for all $k = 1, 2, \ldots, k^*$ and $T^{k^*} = \emptyset$. For any nonempty $T' \subseteq T$, there exists some k such that $T' \subseteq T^k$ and $T' \nsubseteq T^{k+1}$. It is straightforward to verify that the operator $\Phi(\cdot)$ is monotonically increasing in the set inclusion sense. As such, $\Phi(T') \subseteq \Phi(T^k) = T^{k+1}$. Since $T' \nsubseteq T^{k+1}$, we have $\Phi(T') \neq T'$. Thus, the data set satisfies the never-covered property under $\trianglerighteq$. □

Proof of Theorem 2. By Theorem 1, Statements (1) and (2) are equivalent, and obviously (3) implies (1). So it remains to show that (2) implies (3).

To the original data set $\mathcal{M}$ we add the observations $\left\{(A^{at}, A^{at})\right\}_{t\in T}$ and $\left\{(B^{bt}, B^{bt})\right\}_{t\in T}$, where $A^{at} = A^t$ and $B^{bt} = B^t$ for each $t \in T$. Consider the augmented data set

$$\mathcal{M}^* = \left\{(A^t, B^t)\right\}_{t\in T} \bigcup \left\{(A^{at}, A^{at})\right\}_{t\in T} \bigcup \left\{(B^{bt}, B^{bt})\right\}_{t\in T}.$$

where all the added observations are considered weak preferences. Notice that $\mathcal{M}$ is nicely rationalizable if and only if $\mathcal{M}^*$ is rationalizable.

We denote a typical observation of $\mathcal{M}^*$ by z; the observation z may be in T, in $aT := \{a1, a2, \ldots, at, \ldots\}$, or in $bT := \{b1, b2, \ldots, bt, \ldots\}$. Suppose that $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ satisfies the never-covered property, i.e., for any $T' \subseteq T$, the set of dominated observations at T', $\Phi(T')$, satisfies $\Phi(T') \neq T'$. We claim that this implies that $\mathcal{M}^*$ also satisfies the never-covered property, which will guarantee (by Theorem 1) that $\mathcal{M}^*$ is rationalizable.

Given Z', a nonempty subset of Z, we denote its dominated observations by $\Phi^*(Z')$. We need to show that $\Phi^*(Z') \neq Z'$. Suppose that $Z' \cap S = \emptyset$, where $S \subseteq T$ is the set of observations with strict preferences. Then $\Phi^*(Z') = \emptyset$ and so obviously $\Phi^*(Z') \neq Z'$. Now suppose that $Z' \cap S$ is nonempty, which means that $Z' \cap T$ is also nonempty. Since $\mathcal{M}$ satisfies the never-covered property, $(Z' \cap T) \setminus \Phi(Z' \cap T)$ is nonempty. It is straightforward to check in this case that if $\hat{t} \in (Z' \cap T) \setminus \Phi(Z' \cap T)$

then $\hat{t} \in Z' \setminus \Phi^*(Z')$. Therefore, $Z' \setminus \Phi^*(Z')$ is nonempty since $(Z' \cap T) \setminus \Phi(Z' \cap T)$ is nonempty. We conclude that $\mathcal{M}^*$ satisfies the never-covered property. $\square$

Menu preference rationalization and upper contour rationalization. It is straightforward to check that menu A is strictly preferred to menu B if and only if there is an upper contour rationalization of the following set of observations:

$$\Big(\{\{y\}\}_{y\in A}, x\Big) \text{ for each } x \in B.$$

For example, $\{x, y\}$ is strictly preferred to $\{z, w\}$ if and only if there is an upper contour rationalization of the observations $(\{\{x\}, \{y\}\}, z)$ and $(\{\{x\}, \{y\}\}, w)$. Conversely, suppose we wish to guarantee that there is a set in the collection $\{A_j\}_{j\in J}$ that is contained in the upper contour set of x; this is equivalent to a rationalization of the following set of menu preference pairs:

$$\Big(\cup_{j\in J}\, \{y_j\}, x\Big) \text{ for each } \cup_{j\in J}\, \{y_j\} \text{ where } y_i \in A_j \text{ for all } j \in J.$$

For example, either $\{x, y\}$ or $\{z\}$ is in the upper contour set of w if and only if there is a rationalization of the following menu preferences: $\{x, z\}$ is preferred to $\{w\}$ and $\{y, z\}$ is preferred to $\{w\}$.

Thus, it is always possible to convert an upper contour rationalization problem into a menu preference rationalization problem and vice versa when X is finite (but not when it is infinite) and any algorithm developed for one problem could, in principle, be used to solve the other. However, it should also be clear from the conversion procedure we outlined above that there is no general reason for solving either problem in this roundabout fashion, since the converted data set would have more (and in some case many more) observations than the original data set. The two algorithms are best understood as distinct and serving different purposes.

Proof of Theorem 3. The equivalence of Statements (1) and (2) follows from Theorem 1 and obviously (3) implies (1). It suffices to show that Statement (2) implies (3). By Theorem 1, since $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, it admits a no-cycling selection $\{x^t\}_{t\in T}$. From the proof of Theorem 1, we know that any preference that extends $\text{tran}(R^* \cup \trianglerighteq)$ (the transitive closure of $R^* \cup \trianglerighteq$) will rationalize the data and extend $\trianglerighteq$. It remains to show that there is a preference representable by a continuous utility function that extends $\text{tran}(R^* \cup \trianglerighteq)$. By Levin's Theorem, such an extension exists so long as $\text{tran}(R^* \cup \trianglerighteq)$ is a closed preorder. That is indeed the case (see the proof of this claim contained in the proof of Theorem 2 in Nishimura et al. (2017)) and it follows from the compactness of B^t for all $t \in T$ and the finite number of observations. $\square$

Proof of Theorem 4. Statement (3) obviously implies (1). To show that (1) implies (2), suppose that $\mathcal{M}$ is rationalized by a locally nonsatiated preference $\succsim$ and $x^t \in L(p^t)$ satisfies $x^t \succsim L(q^t)$ for all $t \in T$ and $x^t \succ L(q^t)$ for all $t \in S$. Let $\hat{x} = \max(\{x^t\}_{t \in T'}; \succsim)$. Denoting $L_p(T') = \bigcup_{t \in T'} L(p^t)$ and $L_q(T')$ in an analogous fashion, we note that $L_p(T')$ cannot be covered by $L_q(T')^{\downarrow\downarrow} \bigcup L_q(T' \cap S)^{\downarrow}$ since the former contains $\hat{x}$ and the latter does not. Indeed, $\hat{x} \succ L_q(T' \cap S)^{\downarrow} = L_q(T' \cap S)$, and so $\hat{x}$ is not in $L_q(T' \cap S)$. Nor can $\hat{x}$ be in $L_q(T')^{\downarrow\downarrow} = L_q(T')^o$ (the interior of $L_q(T')$) because $\hat{x} \succsim L_q(T')$ and the preference is locally nonsatiated. Taking it one step further, we know that $\hat{x}$ is not in $L_q(T')^o \bigcup L_q((T' \cap S) \cup \Phi^1(T'))$, where

$$\begin{aligned} \Phi^1(T') &:= \left\{ t \in T' : L(p^t) \subseteq L_q(T')^{\downarrow\downarrow} \bigcup L_q(T' \cap S)^{\downarrow} \right\} \\ &= \left\{ t \in T' : L(p^t) \subseteq L_q(T')^o \bigcup L_q(T' \cap S) \right\}. \end{aligned}$$

This is because $\hat{x} \succ L_q((T' \cap S) \cup \Phi^1(T'))$. Repeating this argument, we eventually conclude that $\hat{x}$ is in $A(T')$ but not in $L_q(T')^o \bigcup L_q((T' \cap S) \cup \Phi(T'))$ and hence $\Phi(T') \neq T'$.

To show that (2) implies (3), note that Theorem 3 guarantees that there is a strictly increasing and continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that rationalizes $\mathcal{M}$. So it suffices to show that there is a utility function $\hat{u}$ that also rationalizes $\mathcal{M}$, which has the additional property of concavity. Each budget, $L(p^t)$ is compact and so it has an optimum under u which we denote by x^t (if there are multiple optimal alternatives we may pick any one of them); similarly we denote the optimum bundle in $L(q^t)$ by y^t. Since u is strictly increasing, $p^t \cdot \bar{x}^t = q^t \cdot y^t = 1$, i.e., the optimal bundle is on the budget plane and not just the budget set. Let $\succsim$ denote the preference (i.e, the complete preorder) over $\{x^t\}_{t \in T} \cup \{y^t\}_{t \in T}$ induced by u. Since it is generated by u, the notional data set $\mathcal{N} = \{(x^t, p^t)\}_{t \in T} \cup \{(y^t, q^t)\}_{t \in T}$ is cyclically consistent in the sense of Afriat (1967) (equivalently, obeys the generalized axiom of revealed preference in the sense of Varian (1982)). The preference $\succsim$ is a completion of the revealed preference relations generated by $\mathcal{N}$ and defined on $\{x^t\}_{t \in T} \cup \{y^t\}_{t \in T}$; by the Afriat's Theorem, there is a strictly increasing, continuous and *concave* utility function $\hat{u}$ such that $\hat{u}(x^t) \geq \hat{u}(x)$ for all $x \in L(p^t)$, $\hat{u}(y^t) \geq \hat{u}(x)$ for all $x \in L(q^t)$, and $\hat{u}(x^t) \geq (>)\, u(y^t)$ if $x^t \succsim (\succ)\, y^t$ (where $\succ$ is the asymmetric part of $\succsim$). Thus $\hat{u}$ also rationalizes $\mathcal{M}$. $\square$

Proof of Corollary 2. By Theorem 4, it suffices for us to show that $L(p^t) \not\subseteq$

$\bigcup_{t'\in T'} L(q^{t'})$ if and only if $p^t \not\geq q$ for any $q \in conv\Big(\{q^s\}_{s\in T'}\Big)$.

(The only if-part) Suppose that there exists some $q \in \text{conv}\Big(\{q^s\}_{s\in T'}\Big)$ such that $p^t \geq q$. Then for any $x \in L(p^t)$, we have $p^t \cdot x \geq q \cdot x$, which implies $p^t \cdot x \geq q^{t'} \cdot x$ for some $t' \in T'$. Thus, $x \in \bigcup_{s\in T'} L(q^s)$. Since this is true for all any $x \in L(p^t)$, we have $L(p^t) \subseteq \bigcup_{s\in T'} L(q^s)$.

(The if-part) Fix some nonempty $T' \subseteq T$. Let

$$Q := \Big\{p \in \mathbb{R}^n_+ : p \geq q \text{ for some } q \in \text{conv}\Big(\{q^s\}_{s\in T'}\Big)\Big\}.$$

Clearly, Q is closed and convex. Since there exists some $t \in T'$ such that $p^t \notin Q$, by the Hyperplane Separating Theorem, there exists a vector $r \in \mathbb{R}^n$ and a number b where $r \neq 0$ such that $p^t \cdot r = b < q \cdot r$ for any $q \in Q$. It is easy to verify that $r \geq 0$. Since $r \neq 0$, we have $b = p^t \cdot r > 0$. Let $r' = \frac{r}{b}$. We have $p^t \cdot r' = 1 < q \cdot r'$ for all $q \in Q$. In words, r' is affordable under the price vectore p^t but not under any q^s. Therefore, $L(p^t) \not\subseteq \bigcup_{s\in T'} L(q^s)$. $\square$

The next result extends Corollary 2 to the case in which weak preference between prices are observed, i.e., where W can be nonempty.

Corollary 4. *Consider a set of preferences over budget sets $\mathcal{M} = \Big\{(L(p^t), L(q^t))\Big\}_{t\in T}$. The data set $\mathcal{M}$ is rationalizable by a strictly increasing, continuous, and concave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ if and only if the following property holds: for any nonempty $T' \subseteq T$,*

> *there exists $\Phi \subsetneq T'$ and $\epsilon \gg 0$ such that $p \not\geq q$ whenever $p \ll p^t$ for some $t \in T' \setminus \Phi$ and $q \in conv\Big(\{q^s\}_{s\in T'} \bigcup \{q^r - \epsilon\}_{r\in\Phi\cup(T'\cap S)}\Big)$.*[28]

Proof of Corollary 4. By Theorem 4, it suffices to show that $\mathcal{M}$ satisfies the never-covered property under $\geq$ if and only if the property in Corollary 4 holds. The never-covered property under $\geq$ requires that for any nonempty $T' \subseteq T$, $T' \setminus \Phi(T') \neq \emptyset$. By the definition of $\Phi(T')$, if $t \in T' \setminus \Phi(T')$, then

$$L(p^t) \not\subseteq \left(\bigcup_{t'\in T'} L(q^{t'})^o\right) \bigcup \left(\bigcup_{t'\in(T'\cap S)\cup\Phi(T')} L(q^{t'})\right).$$

We first prove the following lemma.

[28] When $T = S$, the characterizing condition in this corollary reduces to that in Corollary 2. To see this, note that if $T' \cap S = T'$, then the condition in Corollary 4 is equivalent to saying that for any nonempty $T' \subseteq T$, there exists $t \in T'$ and $\epsilon \gg 0$ such that for any $p \ll p^t$ and $q \in \text{conv}\big(\{q^s\}_{s\in T'} \bigcup \{q^r - \epsilon\}_{r\in T'}\big)$, $p \not\geq q$. By continuity, this is equivalent to the condition in Corollary 2.

Lemma 2. *$L(p) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$ if and only if there exists some $\epsilon \gg 0$ such that for any $\hat{p} \ll p$, $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$.*

Proof. (The only if-part) Since $L(p) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$, there exists some $x \in L(p)$ such that $x \notin \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$. Fix such an x. Since $x \in L(p)$, for any $\hat{p} \ll p$, there exists some $\delta \gg 0$ such that $x + \delta \in L(\hat{p})$. Fix such a δ. Since $x \notin \bigcup_{s=1}^{n} L(q^s)$, there exists some sufficiently small $\epsilon \gg 0$ such that $x \notin \bigcup_{s=1}^{n} L(q^s - \epsilon)$, which further implies that $x + \delta \notin \bigcup_{s=1}^{n} L(q^s - \epsilon)$. Since $x \notin \bigcup_{r=1}^{m} L(q^r)^o$, $x + \delta \notin \bigcup_{r=1}^{m} L(q^r)$. Since $x + \delta$ is contained in $L(\hat{p})$ but not in $\left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$, we have $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$.

(The if-part) Fix $\epsilon \gg 0$ such that for any $\hat{p} \ll p$, $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$. Clearly, for each $\theta \in (\frac{1}{2}, 1)$, $\theta p \ll p$, and there exists $x^\theta \in L(\theta p)$ such that $x^\theta \notin \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$. Note that $\{x^\theta\}_{\theta \in (\frac{1}{2}, 1)} \in L(\frac{1}{2}p)$, which is compact. Therefore, $\{x^\theta\}_{\theta \in (\frac{1}{2}, 1)}$ has a convergent subsequence. Let x denote the limit of this subsequence. Since $x^\theta \in L(\theta p)$, $x \in L(p)$. Furthermore, (1) $x \notin \left(\bigcup_{s=1}^{n} L(q^s)\right)$, otherwise $x \in L(q^s - \epsilon)^o$ for some s, which implies that $x^\theta \in L(q^s - \epsilon)^o$ for some θ and some s; and (2) $x \notin \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$, otherwise $x^\theta \in L(q^r)$ for some r and some θ. Since x is contained in $L(p)$ but not in $\left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$, we have $L(p) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$. □

It is easy to see that $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$ if and only if for all $q \in conv\left(\{q^s - \epsilon\}_{s=1}^{n} \cup \{q^r\}_{r=1}^{m}\right)$, $\hat{p} \ngeq q$ (using similar argument as in the proof of the if-part of Corollary 2). The desired result then follows from this and Proposition 1. □

Proof of Theorem 5. We skip the proof that Statement (1) implies (2), which is straightforward and similar to the argument given in Section 4 for the claim that (1) implies (2) in Theorem 4. It is also obvious that (3) implies (1). So it remains for us to show that (2) implies (3). An appeal to Theorem 3 guarantees that there is x^t (for each $t \in T$) and a strictly increasing and continuous utility function $\tilde{u} : \mathbb{R}^n_+ \to \mathbb{R}$ such that $\tilde{u}(x^t) \geq \tilde{u}(x)$ for all $x \in L(p^t)$. Therefore the notional data set $\left\{(x^t, p^t)\right\}_{t \in T}$ must satisfy cyclical consistency (equivalently GARP). By Afriat's Theorem, there is a strictly increasing, continuous, and concave utility function u such that $u(x^t) \geq u(x)$ for all $x \in L(p^t)$ for all $t \in T$. □

Proof of Theorem 6. Statement (1) ⇒ Statement (2): Suppose that the data set (Σ, f) is rationalizable by multiple preferences. Fix an arbitrary nonempty $\Sigma' \subseteq \Sigma$

and $B \in \Sigma$. Since (Σ, f) is rationalizable by multiple preferences, $g(C) \subseteq g\Big(\cup_{A\in\Sigma'} A\Big)$ for all $C \in \Sigma'$. It follows that $\cup_{A\in\Sigma'} g(A) \subseteq g\Big(\cup_{A\in\Sigma'} A\Big)$. We then have

$$\begin{aligned} f\Big(\cup_{A\in\Sigma'} A\Big) &= \cup_{A\in\Sigma'} A \setminus g\Big(\cup_{A\in\Sigma'} A\Big) \\ &\subseteq \cup_{A\in\Sigma'} A \setminus \cup_{A\in\Sigma'} g(A) \\ &= \cup_{A\in\Sigma'} (f(A) \cup g(A)) \setminus \cup_{A\in\Sigma'} g(A) \\ &= \cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A). \end{aligned}$$

Thus, if $\Big(\cup_{A\in\Sigma'} f(A)\setminus\cup_{A\in\Sigma'} g(A)\Big) \subseteq B$, we must have that $f\Big(\cup_{A\in\Sigma'} A\Big) \subseteq B$. Since (Σ, f) is rationalizable by multiple preferences and $f\Big(\cup_{A\in\Sigma'} A\Big) \subseteq B$, $f(B)\cap g\Big(\cup_{A\in\Sigma'} A\Big) = \emptyset$. Since $\cup_{A\in\Sigma'} g(A) \subseteq g\Big(\cup_{A\in\Sigma'} A\Big)$, we have $f(B) \cap \Big(\cup_{A\in\Sigma'} g(A)\Big) = \emptyset$.

Statement (2) $\Rightarrow$ Statement (3): We show that for each $A \in \Sigma$ and $x \in f(A)$, the data set $\mathcal{O}_{A,x}$ is rationalizable by a strict preference. Suppose to the contrary, for some $A \in \Sigma$ and $x \in f(A)$, the data set $\mathcal{O}_{A,x}$ is not rationalizable by a strict preference. By Corollary 1, there exists some nonempty $\Sigma' \subseteq \Sigma$ such that

$$\cup_{B\in\Sigma'} f_{A,x}(B) \subseteq \cup_{B\in\Sigma'} g_{A,x}(B). \tag{5}$$

Recall that in the main text, we show that the second statement implies that the data set $\mathcal{O} = \Big\{(f(A), g(A))\Big\}_{A\in\Sigma}$ is rationalizable by a strict preference. By Corollary 1,

$$\cup_{B\in\Sigma'} f(B) \nsubseteq \cup_{B\in\Sigma'} g(B.)$$

Since $\mathcal{O}$ and $\mathcal{O}_{A,x}$ only differ when the menu is A, we can conclude that $A \in \Sigma'$. Let $\Sigma'' := \Sigma' \setminus \{A\}$. Since $\mathcal{O} = \Big\{(f(A), g(A))\Big\}_{A\in\Sigma}$ is rationalizable by a strict preference, by Corollary 1, we have

$$\cup_{B\in\Sigma''} f_{A,x}(B) \nsubseteq \cup_{B\in\Sigma''} g_{A,x}(B). \tag{6}$$

It follows from (5) and (6) that $\Big(\cup_{B\in\Sigma''} f_{A,x}(B) \setminus \cup_{B\in\Sigma''} g_{A,x}(B)\Big) \subseteq g_{A,x}(A) \subseteq A$ and $x \in \cup_{B\in\Sigma''} g_{A,x}(B)$. Therefore, $x \in f(A) \cap \Big(\cup_{B\in\Sigma''} g_{x,A}(B)\Big)$. Since $\mathcal{O}$ and $\mathcal{O}_{A,x}$ coincide on Σ'', the second statement requires that $f(A) \cap \Big(\cup_{B\in\Sigma''} g_{x,A}(B)\Big) = \emptyset$. We arrive at a contradiction.

Statement (3) $\Rightarrow$ Statement (1): Suppose that for any $A \in \Sigma$ and $x \in f(A)$, $\mathcal{O}_{A,x}$ is rationalizable by a strict preference. Let $\succ_{A,x}$ denote a strict preference that rationalizes $\mathcal{O}_{A,x}$. By definition, $\max(A; \succ_{A,x}) = x \in f(A)$ and $\max(A'; \succ_{A,x}) \in f(A')$ for any $A' \in \Sigma$ and $A' \neq A$. We claim that the set of strict preferences

$\{\succ_{A,x}\}_{A\in\Sigma,\,x\in f(A)}$ rationalizes (Σ, f) under the multiple preferences model. It suffices to show that

$$f_{\{\succ_{A,x}\}_{A\in\Sigma,\,x\in f(A)}}(A') = f(A')$$

for all $A' \in \Sigma$. This follows immediately from the construction of the set of strict preferences, since for each $A' \in \Sigma$, (1) each element y in $f(A')$ is the maximal element in A' according to the strict preference $\succ_{A',y}$; and (2) for any strict preference $\succ$ in $\{\succ_{A,x}\}_{A\in\Sigma,\,x\in f(A)}$, the maximal element in A' according to $\succ$ lies in $f(A')$. $\square$

Proof of Corollary 3. The only if-part is trivial. We only prove the if-part below. We show that the second statement in Theorem 6 follows from the Chernoff axiom and the Aizerman axiom.

Claim 1. *For any $A \in \mathcal{X}$, if $x \in g(A)$, then $f(f(A) \cup x) = f(A)$.*

Proof. Since $\big(f(A) \cup x\big) \subseteq A$, the Chernoff axiom implies that $f(A) \subseteq f(f(A) \cup x)$. Since $f(A) \subseteq \big(f(A) \cup x\big) \subseteq A$, the Aizerman axiom implies that $f(f(A) \cup x) \subseteq f(A)$. Thus, $f(f(A) \cup x) = f(A)$.

Claim 2. *For any $A, B \in \mathcal{X}$, if $f(A) \subseteq B$, then $f(B) \cap g(A) = \emptyset$.*

Proof. Suppose to the contrary, there exists some $x \in \big(f(B) \cap g(A)\big)$. Since $x \in g(A)$, by Claim 1, x is not chosen in $f(A) \cup x$. The Chernoff axiom implies that x is not chosen in any superset of $f(A) \cup x$. In particular, $x \notin f(B)$. We have a contradiction.

Claim 3. *For any nonempty $\mathcal{X}' \subseteq \mathcal{X}$ and $B \in \mathcal{X}$, if $\big(\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A)\big) \subseteq B$, then $f(B) \cap \big(\cup_{A\in\mathcal{X}'} g(A)\big) = \emptyset$.*

Proof. Fix an arbitrary nonempty $\mathcal{X}' \subseteq \mathcal{X}$ and $B \in \mathcal{X}$. By the Chernoff axiom, $g(C) \subseteq g\big(\cup_{A\in\mathcal{X}'} A\big)$ for all $C \in \mathcal{X}'$. Therefore, $\cup_{A\in\mathcal{X}'} g(A) \subseteq g\big(\cup_{A\in\mathcal{X}'} A\big)$. We then have

$$\begin{aligned}
f\Big(\cup_{A\in\mathcal{X}'} A\Big) &= \cup_{A\in\mathcal{X}'} A \setminus g\Big(\cup_{A\in\mathcal{X}'} A\Big) \\
&\subseteq \cup_{A\in\mathcal{X}'} A \setminus \cup_{A\in\mathcal{X}'} g(A) \\
&= \cup_{A\in\mathcal{X}'} (f(A) \cup g(A)) \setminus \cup_{A\in\mathcal{X}'} g(A) \\
&= \cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A).
\end{aligned}$$

Thus, if $\big(\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A)\big) \subseteq B$, we must have $f\big(\cup_{A\in\mathcal{X}'} A\big) \subseteq B$. By Claim 2, $f(B) \cap g\big(\cup_{A\in\mathcal{X}'} A\big) = \emptyset$. The claim follows since $\cup_{A\in\mathcal{X}'} g(A) \subseteq g\big(\cup_{A\in\mathcal{X}'} A\big)$.

By Theorem 6, the data set is rationalizable by multiple preferences. $\square$

Proof of Theorem 7. Let $\bar{X} := \Big\{(x,y) \in X \times X : x \neq y\Big\}$.

(The only if-part) Suppose that the data set (Σ, f) is rationalizable under the minimax regret model. By definition, there exists a finite set U of utility functions such that

$$\max_{y \in A \setminus f(A)} \phi_U(f(A), y) < \max_{y \in A \setminus x} \phi_U(x, y)$$

for all $A \in \Sigma$ and $x \in A \setminus f(A)$. Since X is finite and these inequalities are strict, there exists a finite set U' of utility functions such that

$$\max_{y \in A \setminus f(A)} \phi_{U'}(f(A), y) < \max_{y \in A \setminus x} \phi_{U'}(x, y)$$

for all $A \in \Sigma$ and $x \in A \setminus f(A)$, and $\phi_{U'}(x,y) = \phi_{U'}(z,w)$ only if $(x,y) = (z,w)$. Define a strict partial order P such that $(x,y)\, P\, (z,w)$ if and only if $\phi_{U'}(x,y) > \phi_{U'}(z,w)$. It follows from the Szpilrajn's extension theorem that we can extend P to a strict preference $\succ$. It is straightforward to verify that $\bar{\mathcal{O}}$ can be rationalized by the strict preference $\succ$.

(The if-part) Suppose that $\bar{\mathcal{O}}$ is rationalizable by a strict preference, say $\succ$. It suffices to show that there exists a finite set U of utility functions such that for any $(x,y), (z,w) \in \bar{X}$, $\phi_U(x,y) > \phi_U(z,w)$ if and only if $(x,y) \succ (z,w)$.

Since $\bar{X}$ is finite, we can construct a function $\beta : \bar{X} \to (1,2)$ such that $\beta(x,y) > \beta(z,w)$ if and only if $(x,y) \succ (z,w)$. We now construct a finite set of utility functions

$$U = \{u_{x,y}\}_{(x,y) \in \bar{X}},$$

where the utility functions are indexed by $(x,y) \in \bar{X}$. For each $(x,y) \in \bar{X}$, let

$$u_{x,y}(z) = \begin{cases} 0, & \text{if } z = x; \\ \beta(x,y), & \text{if } z = y; \\ \frac{\beta(x,y)}{2}, & \text{otherwise.} \end{cases}$$

Obviously, $u_{x,y}(y) - u_{x,y}(x) = \beta(x,y) > 1$. We claim that $u_{z,w}(y) - u_{z,w}(x) < 1$ if $(z,w) \neq (x,y)$. First consider the case in which $w \neq y$. It follows from the construction of the utility functions and the β function that

$$u_{z,w}(y) - u_{z,w}(x) \leq u_{z,w}(y) \leq \frac{\beta(z,w)}{2} < 1.$$

Next, we consider the case in which $w = y$. Since $(z,w) \neq (x,y)$, we have $z \neq x$. By the construction of the utility functions,

$$u_{z,w}(y) - u_{z,w}(x) = \beta(z,w) - \frac{\beta(z,w)}{2} = \frac{\beta(z,w)}{2} < 1.$$

Therefore, we can conclude that

$$\phi_U(x,y) = \max_{u \in U} \Big\{u(y) - u(x)\Big\} = u_{x,y}(y) - u_{x,y}(x) = \beta(x,y).$$

We have constructed a finite set U of utility functions such that for any $(x,y),(z,w) \in \bar{X}$,

$$\phi_U(x,y) > \phi_U(z,w) \iff \beta(x,y) > \beta(z,w) \iff (x,y) \succ (z,w).$$

This completes the proof of the if-part. □

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