Optimal Multi-unit Allocation with Costly Verification

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Abstract

A principal has n homogeneous objects to allocate to $I > n$ agents. The principal can allocate at most one good to an agent and each agent values the good. Agents have private information about the principal's payoff of allocating the goods. There are no monetary transfers but the principal can costly check any agent's value. We characterize the direct mechanism which maximizes the principal's net expected payoff. Such an optimal mechanism is easily implementable by a dynamic game which has an equilibrium in obviously dominant strategies.

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1 Introduction

Consider a principal who allocates n identical goods to $I > n$ agents. The principal can allocate at most one good to an agent and the agents strictly prefer to receive a good than not receiving one. Each agent i has private information about the value to the principal, denoted by t_i, if he receives the good. The principal cannot incentivize the agents to report their private information through monetary transfers; however, she is able to check the value of each agent at a cost c_i that may vary across agents. Many examples from various industries and organizations fit the environment described above. A humanitarian organization, such as IFRC (International Federation of Red Cross and Red Crescent Societies), may need to decide which region to support after a disaster, e.g. in which town(s) to rebuild a hospital or a primary school. The ministry of education may need to decide to which research teams to allocate the research funds such as the Academic Research Fund (AcRF) in Singapore. A dean may have several job openings to allocate to different departments in the university. A chief procurement officer (CPO) and his team may decide which suppliers to invest for sustainability programs.

In this paper, we construct a class of mechanisms, the n-ascending mechanisms (n-ASMs), that maximize the expected net value to the principal, that is, the total value from allocating the goods minus the checking costs. Such an optimal mechanism is a hybrid between two extremes. One extreme allocates the goods efficiently, i.e., to the n agents who have the highest (net) values, which requires agents being checked and leads to very high checking cost; the other incurs no checking cost by allocating the goods without learning agents' values, which results in very poor allocation efficiency. The n-ASMs balance the benefit of efficient allocation and the cost of checking agents. Moreover, the trade-off between the two extremes is determined ex post, in response to each value profile $\mathbf{t} = (t_1, \ldots, t_I)$. See Theorem 1.

More precisely, a n-ASM specifies n agents, e.g. $i = 1, \ldots, n$, and a threshold v_i^* for each agent such that $v_1^* \geq v_2^* \geq \cdots \geq v_n^*$.[1] These agents are referred to as the favored agents, and they are "endowed" with the goods. The remaining agents are the free agents. The n-ASM specifies for each value profile an allocation rule and a checking policy that can be determined by at most n steps. In step 1, if no free agent reports a net value $t_i - c_i$ higher than the smallest threshold v_n^*, then allocate all goods to the n favored agents without checking anyone and the allocation process ends. Otherwise, check the agent with the highest reported net value among the free agents and agent n, and allocate a good to that agent. In equilibrium, all agents report truthfully and hence the allocation rules are consistent with the checking outcomes. We remove the agent who has obtained the good, and the system is left

[1] The agents $i = 1, \ldots, n$ may not be the default first n agents, i.e. we may relabel agents according to their value distributions. See Section 3.1.

with $I-1$ agents and $n-1$ goods. Agent n becomes a free agent if he is not removed. Then, we proceed to step 2 which is very similar to step 1 but with the threshold v^*_{n-1} and the updated set of free agents. In Section 3.3, we argue that an extensive form game based on the process above has an equilibrium in obviously dominant strategies (Li, 2017), i.e. the n-ASM has an obviously strategy-proof (OSP) implementation. See Theorem 2.

The rest of this section reviews the literature. Section 2 introduces the model. Section 3 defines the n-ASM and presents our results on its optimality and OSP implementation. In Section 4, we sketch the proof of the optimality result. Section 5 concludes.

The Related Literature Townsend (1979) initiated the literature on the principal-agent model with costly state verification.[2] Recently, Ben-Porath et al. (2014) extend the costly state verification framework to allow for multiple agents when monetary transfers are not possible.[3] Particularly, they study the allocation of one good to multiple agents, which is a special case of our paper. Thus, our optimal mechanism, the n-ASM, reduces to their optimal favored-agent mechanism when $n = 1$.

In the multi-unit setting, our paper is most closely related to Ben-Porath et al. (2019), which examines mechanism design with evidence in a very general environment.[4] The evidence model differs significantly from the costly-verification model. However, Ben-Porath et al. (2019) show that a large class of costly-verification models can be studied using the equilibrium analysis of an associated model of evidence. More precisely, their approach can be used to solve the multi-unit allocation problem as we do but in a model with finitely many type profiles.[5] It can also be used to characterize optimal mechanisms for the model of Erlanson and Kleiner (2020), which studies costly verification in collective decisions. We draw two more detailed connections with Ben-Porath et al. (2019) in what follows.

First, focusing on the costly-verification aspect, there is only one modeling difference between two papers: Ben-Porath et al. (2019) assume that each agent's value is distributed over finitely many types, while we assume that each agent's value is distributed over an interval. In the discrete-type setting, Ben-Porath et al. (2019) build the mathematical link between the costly-verification model and the evidence model by changing variables. Using their results in the evidence model, one can easily derive an optimal allocation rule for the costly-verification problem, which in turn can induce an optimal (reduced-form) checking policy. Unsurprisingly, their allocation rule and (reduced-form) checking policy have exactly the same formats as those of our n-ASM. In this sense, our paper serves as an extension of

[2] See also Gale and Hellwig (1985), Border and Sobel (1987), and Mookherjee and Png (1989). Our work is also related to, among others, Glazer and Rubinstein (2004, 2006), Green and Laffont (1986), Bull and Watson (2007), Deneckere and Severinov (2008), Ben-Porath and Lipman (2012), Kartik and Tercieux (2012), Sher and Vohra (2015) and Doval (2018) in the same sense as Ben-Porath et al. (2014); we refer the readers to their paper.

[3] See also Lipman (2015) and Erlanson and Kleiner (2019).

[4] See Dye (1985) and Jung and Kwon (1988) for earlier work on the evidence models.

[5] See Section 3.2 and Appendix C of Ben-Porath et al. (2019) for how their approach works when there are only finitely many types. Our approach for the continuous-type model differs significantly from theirs. See Section 4.

Ben-Porath et al. (2019), from the discrete-type setting to the continuous-type setting. Moreover, our analysis complements theirs in specifying the *ex post*, instead of the reduced-form, checking policies.[6]

Second, we provide an OSP implementation of the optimal mechanism in both the continuous-type setting and the discrete-type setting of Ben-Porath et al. (2019). Namely, to implement an optimal mechanism, the principal can first "endow" the n favored agents each with a good, and then allow those agents who do not have a good to "challenge" a favored agent who is still holding an endowment. At each round of challenging, the principal only checks the agent who reports the highest net value among those agents involved. Such an OSP implementation is new compared with Ben-Porath et al. (2019).

The literature on state verification grows rapidly in recent years. In addition to those discussed above, Mylovanov and Zapechelnyuk (2017) study the allocation of an indivisible good with ex post costless verification and limited punishment. More precisely, in Ben-Porath et al. (2014) as well as the current paper, once the agents are found lying, the principal can punish them by allocating nothing to them. This is seen as unlimited punishment. In contrast, agents in Mylovanov and Zapechelnyuk (2017) may still get some benefit even if they are found lying. Li (2020) investigates a single-good allocation problem with costly verification as in Ben-Porath et al. (2014), but with limited punishment as in Mylovanov and Zapechelnyuk (2017).[7] Erlanson and Kleiner (2020) study how a principal should optimally choose between implementing a new policy and maintaining the status quo when information relevant for the decision is privately held by agents. Again, there is no monetary transfers but costly verification is possible. Epitropou and Vohra (2019) consider a single-good allocation problem where agents arrive on-line. It is a dynamic mechanism design problem where the decision to allocate the good to an agent must be made upon his arrival and is irreversible.

2 The Model

Preliminaries. The set of agents is $\mathcal{I} = \{1, 2, \ldots, I\}$. There are n identical indivisible goods to be allocated among the agents, where $1 \leq n \leq I - 1$. Each agent can receive at most one good. The value to the principal of assigning one good to agent i is t_i, which is i's private type. We normalize so that types are always nonnegative and the value of retaining a good is zero. Assume that the t_i's are independently distributed. For each i, the distribution of t_i has a strictly positive density f_i over the interval $T_i = [\underline{t}_i, \bar{t}_i]$ where $0 \leq \underline{t}_i < \bar{t}_i < \infty$. We use F_i to denote the corresponding distribution function and F to denote the joint distribution. Let $\mathbf{T} = \prod_i T_i$ be the set of type profiles.

The principal can *check* the type of agent i at a cost $c_i > 0$. We interpret checking as obtaining information (e.g. by requesting documentation, interviewing the agent, or hiring outside evaluators)

[6] See Remark 3 in Section 3.2 for more detailed discussions.

[7] See also Li (2021) for the allocation of goods to financially constrained agents.

which perfectly reveals the type of the agent being checked. The cost to the agent of providing information is assumed to be zero.

We assume that every agent strictly prefers receiving a good to not receiving one. Consequently, we can take the payoff to an agent to be the probability with which he receives a good. The intensity of the agents' preferences plays no role in the analysis, so these intensities may or may not be related to the types. We also assume that each agent's reservation utility is less than or equal to his utility from not receiving a good. Since monetary transfers are not allowed, not receiving a good delivers the worst payoff to an agent. Consequently, individual rationality constraints do not bind and so are disregarded throughout.

Direct Mechanisms. A general mechanism in our setting is a game that specifies (i) how each agent may act, (ii) which agents are checked (and when), and (iii) depending on (i) and (ii), who receive the goods. We can apply the same argument as in the online appendix of Ben-Porath et al. (2014), up to terminology replacement, to establish a revelation principle for the current setting. Hence, we restrict our attention to direct mechanisms.

Formally, a *direct mechanism* consists of (i) a checking policy that maps each reported type profile $\mathbf{t} \in \mathbf{T}$ to a profile of checking probabilities $\mathbf{q}(\mathbf{t}) := (q_1(\mathbf{t}), \ldots, q_I(\mathbf{t}))$, and (ii) an allocation rule that assigns the goods to the agents according to their reported type profile and the checking outcomes. Specifically, the allocation depends on whether or not each agent i is checked, and if checked, whether he is truth-telling or lying. If agent i is not checked, we let $p_i^{nc}(\mathbf{t})$ denote the probability of assigning a good to agent i conditional on no checking. As a result, $(1 - q_i(\mathbf{t}))p_i^{nc}(\mathbf{t})$ is the probability of agent i receiving a good without being checked.

If agent i is checked, then there may be two outcomes: truth-telling or lying. To provide the incentive for agent i to tell the truth (in an optimal direct mechanism), the probability of assigning a good to him conditional on checking and lying has to be zero.[8] Since checking is costly, the assignment probability conditional on checking and truth-telling has to be 1; otherwise, the principal can reduce the checking cost by only checking agent i if she decides to assign a good to him (conditional on truth-telling). Therefore, $q_i(\mathbf{t})$ is the probability of checking and assigning a good to agent i. Thus, the total probability of assigning a good to agent i, assuming that he is truth-telling, is

$$p_i(\mathbf{t}) := q_i(\mathbf{t}) + (1 - q_i(\mathbf{t}))p_i^{nc}(\mathbf{t}).$$

Since $(p_i(\mathbf{t}), q_i(\mathbf{t}))$ can be derived from $(p_i^{nc}(\mathbf{t}), q_i(\mathbf{t}))$ and vice versa, we refer to a tuple of functions $(\mathbf{p}, \mathbf{q})$ as a simplified direct mechanism, where $\mathbf{p}(\mathbf{t}) = (p_1(\mathbf{t}), \ldots, p_I(\mathbf{t}))$.

The Principal's Problem. The principal selects a mechanism $(\mathbf{p}, \mathbf{q})$ to maximize her expected net

[8] See part 1 of the online appendix of Ben-Porath et al. (2014) for more detailed discussions.

payoff:

$$\max_{\mathbf{p},\mathbf{q}} \quad \mathbf{E}_t\left[\sum_i \left[p_i(\mathbf{t})t_i - q_i(\mathbf{t})c_i\right]\right] \tag{1}$$

$$\text{subject to} \quad p_i(\mathbf{t}) \in [0,1] \text{ and } q_i(\mathbf{t}) \in [0,1], \quad \forall \mathbf{t} \in \mathbf{T}, \quad \forall i \in \mathcal{I}, \tag{2}$$

$$q_i(\mathbf{t}) \le p_i(\mathbf{t}), \quad \forall \mathbf{t} \in \mathbf{T}, \quad \forall i \in \mathcal{I}, \tag{3}$$

$$\sum_i p_i(\mathbf{t}) \le n, \quad \forall \mathbf{t} \in \mathbf{T}, \quad \text{and} \tag{4}$$

$$\mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i,\mathbf{t}_{-i})\right] \ge \mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i',\mathbf{t}_{-i}) - q_i(t_i',\mathbf{t}_{-i})\right], \quad \forall t_i, t_i' \in T_i, \quad \forall i \in \mathcal{I}. \tag{5}$$

Constraints (2)-(4) are the feasibility constraints. Constraint (5) is the incentive compatibility constraint to ensure that agents always prefer truth-telling to lying.

Given a mechanism $(\mathbf{p},\mathbf{q})$, we define $\hat{p}_i(t_i) := \mathbf{E}_{\mathbf{t}_{-i}} p_i(t_i,\mathbf{t}_{-i})$ and $\hat{q}_i(t_i) := \mathbf{E}_{\mathbf{t}_{-i}} q_i(t_i,\mathbf{t}_{-i})$ for each i. The $2I$ tuple of functions $(\hat{\mathbf{p}},\hat{\mathbf{q}})$ is the *reduced form* of the mechanism $(\mathbf{p},\mathbf{q})$. We say that two mechanisms $(\mathbf{p}^1,\mathbf{q}^1)$ and $(\mathbf{p}^2,\mathbf{q}^2)$ are *equivalent* if they have essentially the same reduced form, i.e. for each $i \in \mathcal{I}$, $\hat{p}_i^1(t_i) = \hat{p}_i^2(t_i)$ and $\hat{q}_i^1(t_i) = \hat{q}_i^2(t_i)$ for almost all $t_i \in T_i$.[9] It is easy to see that we can write the incentive compatibility constraints and the objective function of the principal only in terms of the reduced form of the rules. Hence if $(\mathbf{p}^1,\mathbf{q}^1)$ is an optimal incentive compatible mechanism, $(\mathbf{p}^2,\mathbf{q}^2)$ must be as well. In this paper, we only identify the optimal mechanism up to such equivalence.

3 Characterization of the Optimal Mechanisms

In this section, we define the n-ascending mechanism and present our results on its optimality and obviously strategy-proof implementation. We provide a formal definition of the n-ascending mechanism in Section 3.1. Then, in Section 3.2, we characterize the optimal direct mechanisms by essential randomizations over the n-ascending mechanisms. Finally, Section 3.3 discusses the obviously strategy-proof implementation of the optimal direct mechanism in an extensive form game.

3.1 Definition of the n-Ascending Mechanisms

First of all, let us introduce a threshold v_i^* for each agent $i \in \mathcal{I}$. It is defined as a variant of the critical value t_i^* in Ben-Porath et al. (2014):

$$\mathbf{E}\left(t_i\right) = \mathbf{E}\left(\max\left\{t_i - c_i, v_i^*\right\}\right).^{10} \tag{6}$$

Particularly, the threshold v_i^* is determined by the distribution F_i and the checking cost c_i of agent i, which is simple in the sense that it is independent of (i) the number of agents, (ii) the number of goods, and (iii) the value distributions and checking costs of other agents.

Our interpretation of v_i^* is as follows. Suppose agent i is "endowed" with a good ex ante regardless

[9] The word "essential" means "up to measure-zero subsets of $\mathbf{T}$." This applies throughout the paper.

[10] Ben-Porath et al. (2014) defines a critical value t_i^* implicitly by $\mathbf{E}\left(t_i\right) = \mathbf{E}\left(\max\left\{t_i, t_i^*\right\}\right) - c_i$. It is straightforward to see that $v_i^* = t_i^* - c_i$.

of his value. Suppose also that the principal is not committed to such an allocation and, at this point, considers reallocating the good to pursue potentially higher ex post value. If she maintains the ex ante "endowment," her expected payoff is $\mathbf{E}(t_i)$. If she chooses to trigger reallocation based on ex post values, then there will be two possible outcomes. First, the good still goes to agent i, in which case the principal needs to acquire i's type by checking him and obtains a net value $t_i - c_i$. Second, the good goes to some other agent and brings a net value, say, v to the principal. Obviously, reallocation yields the principal the higher net value between the two, i.e. $\max\{t_i - c_i, v\}$. Note that $\mathbf{E}(\max\{t_i - c_i, v\})$ is increasing in v. Therefore, v_i^* is the minimal net value requested by the principal to trigger reallocation.

This interpretation of v_i^* does not depend on a particular reallocation rule. For example, the reallocation could occur between agent i and the agent who has the *highest* reported net value among $\mathcal{I} \setminus \{i\}$; it could also occur between agent i and the agent who has the *second highest* reported net value among $\mathcal{I} \setminus \{i\}$; etc.[11] Ben-Porath et al. (2014) takes the specific interpretation that the reallocation is between agent i and the agent who has the highest reported net value among $\mathcal{I} \setminus \{i\}$. Our generalized interpretation is particularly useful in the multi-unit setting.

For notational convenience, we rank and relabel agents in the decreasing order of v_i^*, i.e. $v_1^* \geq v_2^* \geq \cdots \geq v_I^*$. The n-ascending mechanism is formally defined below, which is unique given the labelling of agents. It is illustrated in Example 1.

Definition 1. *A mechanism $(\mathbf{p}, \mathbf{q})$ is a **n-ascending mechanism (n-ASM)** if for all $\mathbf{t} \in \mathbf{T}$, the rules $(\mathbf{p}(\mathbf{t}), \mathbf{q}(\mathbf{t}))$ are given by the algorithm below.*

Ascending Algorithm

Set $\mathcal{I}^n = \mathcal{I} \setminus \{1, \ldots, n-1\}$. For each $k = n, \ldots, 1$,

Step $n - k + 1$. *Consider two mutually exclusive cases.*

Case 1. $\max_{i \in \mathcal{I}^k \setminus \{k\}}\{t_i - c_i\} \leq v_k^*$.

Set $p_1(\mathbf{t}) = \cdots = p_k(\mathbf{t}) = 1$, $q_1(\mathbf{t}) = \cdots = q_k(\mathbf{t}) = 0$, and $p_i(\mathbf{t}) = q_i(\mathbf{t}) = 0$ for all $i \in \mathcal{I}^k \setminus \{k\}$. Stop.

Case 2. $\max_{i \in \mathcal{I}^k \setminus \{k\}}\{t_i - c_i\} > v_k^*$.

Define $m := \min\{\arg\max_{i \in \mathcal{I}^k}\{t_i - c_i\}\}$ and set $p_m(\mathbf{t}) = q_m(\mathbf{t}) = 1$.[12]

If $k > 1$, then set $\mathcal{I}^{k-1} := \mathcal{I}^k \cup \{k-1\} \setminus \{m\}$ and go to Step $n - (k-1) + 1$. If $k = 1$, then set $p_i(\mathbf{t}) = q_i(\mathbf{t}) = 0$ for all $i \in \mathcal{I}^k \setminus \{m\}$ and stop.

[11] See Lemma 7 in Section 4 for a case where the reallocation occurs between agent i and the agent who has the *second highest* reported net value among $\mathcal{I} \setminus \{i\}$. For other reallocation rules that are of interest, see Step (ii) of the proof of Proposition 4 in Chua et al. (2019).

[12] Taking the smallest index via $\min\{\cdot\}$ is only for the tie-breaking purpose.

The Ascending Algorithm can be described by the following process. In each step, the first k agents are favored in that they are "endowed" with one good each, while the remaining agents in $\mathcal{I}^k \setminus \{k\}$ are free agents. If no free agent "challenges" the favored agents by reporting a net value higher than v_k^*, then allocate the goods according to the endowment without checking anyone and the allocation process ends. If some free agent challenges the favored agents, then find the agent in $\mathcal{I}^k$ who has the highest net value, check him (the checking outcome will be truth-telling on equilibrium path) and allocate a good to him. At the same time, add agent k to the list of free agents and remove the "successful challenger" who has obtained one good. Then, we proceed with $\mathcal{I}^{k-1}$ and v_{k-1}^*. This continues until all goods are allocated. It is worth emphasizing that the Ascending Algorithm may end at any step between 1 and n, depending on agents' type profile $\mathbf{t}$. The algorithm is illustrated below.

Example 1. *Consider the allocation of two goods among three agents. Suppose $v_1^* > v_2^* > v_3^*$ and $c_1 = c_2 = c_3 = c$. We list three scenarios to illustrate the 2-ASM.*

Scenarios	*Step 1*			*Step 2*			*Good*	*Agents*
	v_k^*	$\mathcal{I}^k$	m	v_k^*	$\mathcal{I}^k$	m	*receivers*	*checked*
$t_1 - c < t_2 - c < t_3 - c < v_2^* < v_1^*$	v_2^*	$\{2,3\}$					$\{1,2\}$	
$t_1 - c < v_2^* < t_2 - c < v_1^* < t_3 - c$	v_2^*	$\{2,3\}$	3	v_1^*	$\{1,2\}$		$\{1,3\}$	$\{3\}$
$v_2^* < v_1^* < t_1 - c < t_2 - c < t_3 - c$	v_2^*	$\{2,3\}$	3	v_1^*	$\{1,2\}$	2	$\{2,3\}$	$\{2,3\}$

In the first scenario, the algorithm ends in one step with case 1 triggered; favored agents 1 and 2 receive the goods and no agent is checked. In the second scenario, the algorithm takes two steps with case 2 and case 1 triggered in order; successful challenger 3 receives a good after being checked and favored agent 1 receives a good without being checked. In the third scenario, the algorithm ends in two steps with case 2 triggered in both steps; agents 2 and 3 receive the goods both as successful challengers.

We provide some intuition for the n-ASM. Given the principal's objective function (1), an optimal mechanism should balance the benefit of efficient allocation and the cost of checking agents. On the one hand, allocating all units according to the "endowments" of agents results in the least checking cost but poor allocation efficiency: the advantage of the reported values are not incorporated. On the other hand, allocating all units based on ex post values, i.e. agents with the n highest net values get the goods, results in high allocation efficiency but high checking cost as well. As such, an optimal mechanism is expected to allocate some goods based on only agents' type distributions while other goods based on their realizations. Taking the "endowments" as a benchmark, as the Ascending Algorithm proceeds, the allocation efficiency improves but the checking cost increases. The algorithm stops at a balanced point of these two effects.

The following observation on the allocation rule of the n-ASM will be useful. It says that given a type profile $\mathbf{t} \in \mathbf{T}$, unless there are ties, the n-ASM allocates the n goods to the n agents who have the n highest values of $w_i := \max\{v_i^*, t_i - c_i\}$.

Lemma 1. *For each $\mathbf{t} \in \mathbf{T}$ and each $i \in \mathcal{I}$, $p_i(\mathbf{t}) = 1$ under the n-ASM if and only if*

$$|\{j \in \mathcal{I} : w_j > w_i\}| + |\{j \in \{1, \dots, i-1\} : w_j = w_i\}| \leq n - 1. \tag{7}$$

3.2 The Characterization Theorem

Theorem 1 below is a full characterization the optimal direct mechanisms for the multi-unit allocation problem, if not unique. One reason for the existence of multiple optimal mechanisms is that whenever a mechanism is optimal, any mechanism that is equivalent to it must be optimal as well. A second reason is that the v_i^*'s may be tied such that, say, $v_1^* = v_2^*$, in which case we have at least two optimal mechanisms according to different labelling of agents. A third reason is that, since the objective function and the constraints in (1)-(5) are all linear in $\mathbf{p}$ and $\mathbf{q}$, any randomization over optimal mechanisms must also be optimal. It turns out that those three are the only reasons for having multiple optimal mechanisms. Theorem 1 says that the n-ASM is the unique optimal mechanism up to equivalence, up to relabelling of agents according to the decreasing order of v_i^*'s and up to randomization (hereafter, equivalence-relabelling-randomization).

Theorem 1. *A mechanism $(\mathbf{p}, \mathbf{q})$ is optimal if and only if it is essentially a randomization over the n-ascending mechanisms.*

We hasten to emphasize that the algorithm-based n-ASM has an equivalent one-shot definition. Note that the Ascending Algorithm which defines the n-ASM potentially takes n steps. However, the key information that the algorithm conveys is merely a cut-off index $k \leq n+1$ between (i) allocation based on the prior value distributions and (ii) allocation based on the ex post realized values. In (i), each agent whose index is smaller than k receives one good without being checked, regardless of their ex post net values; and in (ii), the $n - (k-1)$ agents in $\{k, \dots, I\}$ who have the highest ex post net values each receives one good after being checked. More precisely, given a type profile $\mathbf{t} \in \mathbf{T}$, the cut-off index is $k = n+1$ if $\max_{i>n}\{t_i - c_i\} \leq v_n^*$; otherwise, the cut-off index k is defined as the smallest index such that the number of agents among $\{k+1, \dots, I\}$ who have net values larger than v_k^* is no less than $n - (k-1)$. Formally, $k := \min\{\kappa \in \mathbb{Z}_+ : |\{i > \kappa : t_i - c_i > v_\kappa^*\}| \geq n - (\kappa - 1)\}$. Then, for each $i < k$, $p_i(\mathbf{t}) = 1$ and $q_i(\mathbf{t}) = 0$. For each $i \geq k$,

$$p_i(\mathbf{t}) = q_i(\mathbf{t}) = \begin{cases} 1, & \text{if } |\{l \geq k : t_l - c_l > t_i - c_i\}| + |\{l \in \{k, \dots, i-1\} : t_l - c_l = t_i - c_i\}| \leq n - (k-1) \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1 (Single-good allocation)**.** *We briefly discuss the special case of single-good allocation, i.e. $n = 1$, which is studied in Ben-Porath et al. (2014). Precisely, assuming away ties for simplicity, our n-ASM is as follows: for all $\mathbf{t} \in \mathbf{T}$, if no one other than agent 1 has a net value that is higher than v_1^*, i.e. $k = n + 1 = 2$, then agent 1 obtains the good without being checked, and none of the other agents obtains a good or is checked; if someone other than agent 1 has a net value that is higher than v_1^*, i.e. $k = 1$, then the agent who has the highest net value is checked and allocated the good, and none of the other agents is checked or obtains the good. This is exactly the optimal favored-agent mechanism in Ben-Porath et al. (2014) that favors agent 1 with threshold v_1^*.*

Remark 2 (Strategy-proofness)**.** *Ben-Porath et al. (2014) has argued that the optimal favored-agent mechanism for single-good allocation is ex post incentive compatible. And it can be made strategy-proof by slightly modifying the mechanism off the equilibrium path, without changing the generated allocation and checking cost; see their Footnote 10. With multiple units, the n-ASM preserves these properties.*

Remark 3 (Discrete-type model)**.** *Consider a similar model where $\mathbf{T}$ consists of only finitely many type profiles. Such a discrete-type model can be analyzed by the approach of Ben-Porath et al. (2019). They show that the optimal mechanism has to allocate the n goods to the n agents who have the n highest values of* $\max\{v_i^*, t_i - c_i\}$.[13] *Hence, by Lemma 1, the optimal allocation rules in the discrete-type model and the continuous-type model are identical in format. Once the allocation rules coincide, the* reduced-form *checking policies have to coincide because of the following relation:*

$$\hat{q}_i(t_i) = \hat{p}_i(t_i) - \inf_{t_i' \in T_i} \hat{p}_i(t_i'), \quad \forall t_i \in T_i, \quad \forall i \in \mathcal{I},$$

which says that the reduced form of the optimal checking policy is pinned down by the optimal allocation rule. This is equation (4) in Ben-Porath et al. (2014) and it still holds in our multi-unit case by the same argument. Therefore, in terms of format, the n-ASM is equivalent to the optimal mechanism in the discrete-type model considered by Ben-Porath et al. (2019). The n-ASM complements their optimal mechanism in specifying the ex post*, instead of reduced-form, checking policies.*

3.3 Obviously Strategy-proof Implementation of the n-ASM

The n-ASM is not only an optimal direct mechanism but also contains an obviously strategy-proof implementation in its definition. Particularly, the principal can run an (extensive) *ascending game* according to the Ascending Algorithm, which proceeds as follows:

1. A type profile $\mathbf{t}$ is drawn from the set $\mathbf{T}$ by chance according to the joint distribution function F.
2. Agents in $\mathcal{I}^n \setminus \{n\} = \{n+1, n+2, \ldots, I\}$ simultaneously take actions and each reports a type $t_i' \in T_i$ to the principal.

[13] See their Example 2 and Appendix C for details.

3. The game ends while n favored agents remain if $\max_{i\in\mathcal{I}^n\setminus\{n\}} t'_i - c_i \leq v^*_n$, in which case

   (a) $q_i = 0$ for all $i \in \mathcal{I}$,

   (b) $p_1 = \cdots = p_n = 1$, and $p_i = 0$ for all $i \in \mathcal{I}^{n+1} = \mathcal{I}^n \setminus \{n\}$.[14]

   Otherwise, agent n takes an action by reporting a type $t'_n \in T_n$ to the principal.

4. Let $m_1 := \min\{\arg\max_{i\in\mathcal{I}^n}\{t_i - c_i\}\}$. The game ends while $n-1$ favored agents remain if $\max_{i\in\mathcal{I}^{n-1}\setminus\{n-1\}} t'_i - c_i \leq v^*_{n-1}$ where $\mathcal{I}^{n-1} = \mathcal{I}^n \cup \{n-1\} \setminus \{m_1\}$, in which case

   (a) $q_{m_1} = 1$ and $q_i = 0$ for all $i \in \mathcal{I} \setminus \{m_1\}$,

   (b) $p_{m_1} = 1$ if $t'_{m_1} = t_{m_1}$ and $p_{m_1} = 0$ if $t'_{m_1} \neq t_{m_1}$,

   (c) $p_1 = \cdots = p_{n-1} = 1$, and $p_i = 0$ for all $i \in \mathcal{I}^n \setminus \{m_1\}$.

   Otherwise, agent $n-1$ takes an action by reporting a type $t'_{n-1} \in T_{n-1}$ to the principal.

   ...

n-k+3. Let $m_{n-k} := \min\{\arg\max_{i\in\mathcal{I}^{k+1}}\{t_i - c_i\}\}$. The game ends while k favored agents remain if $\max_{i\in\mathcal{I}^k\setminus\{k\}} t'_i - c_i \leq v^*_k$ where $\mathcal{I}^k = \mathcal{I}^{k+1} \cup \{k\} \setminus \{m_{n-k}\}$, in which case

   (a) $q_j = 1$ for all $j \in \{m_1, \cdots, m_{n-k}\}$ and $q_i = 0$ for all $i \in \mathcal{I} \setminus \{m_1, \cdots, m_{n-k}\}$,

   (b) $p_j = 1$ if $t'_j = t_j$ and $p_j = 0$ if $t'_j \neq t_j$ for all $j \in \{m_1, \cdots, m_{n-k}\}$,

   (c) $p_1 = \cdots = p_k = 1$, and $p_i = 0$ for all $i \in \mathcal{I}^{k+1} \setminus \{m_{n-k}\}$.

   Otherwise, agent k takes an action by reporting a type $t'_k \in T_k$ to the principal.

   ...

n+3. Let $m_n := \min\{\arg\max_{i\in\mathcal{I}^1}\{t_i - c_i\}\}$. The game ends with the following outcome:

   (a) $q_j = 1$ for all $j \in \{m_1, \cdots, m_n\}$ and $q_i = 0$ for all $i \in \mathcal{I} \setminus \{m_1, \cdots, m_n\}$,

   (b) $p_j = 1$ if $t'_j = t_j$ and $p_j = 0$ if $t'_j \neq t_j$ for all $j \in \{m_1, \cdots, m_n\}$, and

   (c) $p_i = 0$ for all $i \in \mathcal{I}^1 \setminus \{m_n\} = \mathcal{I} \setminus \{m_1, \cdots, m_n\}$.

This extensive form game has the features that (i) each agent is called to play at most once (so later on we write strategies as actions instead of as functions of information sets), (ii) a favored agent is called to play only when he is challenged, and (iii) the game ends when there is either no more challenger or no more favored agent. It is straightforward to verify that the outcome $(\mathbf{p}, \mathbf{q})$ of the ascending game starting from type profile $\mathbf{t}$, when agents report their types truthfully, is exactly the $2I$

[14] The outcome of the game is a $2I$ tuple of degenerate probabilities $(\mathbf{p}, \mathbf{q}) \in \{0,1\}^{2I}$. Each agent i cares only about p_i and prefers $p_i = 1$ to $p_i = 0$.

tuple of probabilities $(\mathbf{p}(\mathbf{t}), \mathbf{q}(\mathbf{t}))$ specified by the n-ASM. Compared with the direct mechanism/normal form game which asks all agents to report their types simultaneously and then executes $(\mathbf{p}(\mathbf{t}), \mathbf{q}(\mathbf{t}))$, the ascending game has a notable advantage. Namely, it is "obviously strategy-proof." The concept of obvious strategy-proofness is comprehensively studied by Li (2017) and can be much simplified in our setting.

The full description of a general extensive game form is standard (cf. Li (2017) or pp. 200-201 of Osborne and Rubinstein (1994)), so we only invoke a few key notations. A history is either an empty sequence or a sequence of actions taken by agents (or chance). Let h be a generic history where agents may continue to take actions and z be a generic terminal history. An information set for an agent is a set of histories that are indistinguishable to him. The information sets where agent i is called to play are differentiated merely by t_i, so we let $I_i(t_i)$ be a generic information set. Agent i's strategy specifies an action $t_i' \in T_i$ at each information set $I_i(t_i)$. Let $z(h, t_i', \mathbf{t}'_{-i})$ be the terminal history that results in the ascending game when we start from h and play proceeds according to the action profile $(t_i', \mathbf{t}'_{-i})$ (each agent is called to play at most once). Let $p_i(h, t_i', \mathbf{t}'_{-i}, t_i)$ be the utility (probability of obtaining a good) for agent i if his type is t_i and the terminal history is $z(h, t_i', \mathbf{t}'_{-i})$.

Definition 2. *Given the ascending game and agent i's type t_i, a strategy $t_i' \in T_i$ is **obviously dominant** if at the information set $I_i(t_i)$, for all $t_i'' \in T_i$,*

$$\sup_{h \in I_i(t_i), \mathbf{t}''_{-i} \in \mathbf{T}_{-i}} p_i(h, t_i'', \mathbf{t}''_{-i}, t_i) \le \inf_{h \in I_i(t_i), \mathbf{t}''_{-i} \in \mathbf{T}_{-i}} p_i(h, t_i', \mathbf{t}''_{-i}, t_i)$$

In words, t_i' is obviously dominant if, for any deviation t_i'', at the information set $I_i(t_i)$, the best outcome under the t_i'' is no better than the worst outcome under t_i'. A mechanism is obviously strategy-proof (OSP) if it has an equilibrium in obviously dominant strategies.

As a direct mechanism/normal form game, the n-ASM is not OSP. For example, suppose $t_1 - c_1 > v_1^*$ in the favored-agent mechanism ($n = 1$). Depending on $\mathbf{t}_{-1}$, the best outcome of lying for the favored agent 1 is to still get the good without being checked, when $t_i - c_i < v_1^*$ for all $i > 1$; and the worst outcome of truth-telling is getting no good, when $t_i - c_i > t_1 - c_1 > v_1^*$ for some $i > 1$. Hence, truth-telling is not obviously dominant and the optimal direct mechanism is not OSP. The following theorem says that the ascending game is OSP.

Theorem 2. *The ascending game has a truth-telling equilibrium in obviously dominant strategies.*

Remark 4 (Implementation of the optimal mechanism in the discrete-type model)**.** *The optimal mechanism in the discrete-type model considered by Ben-Porath et al. (2019) can also be implemented by the ascending game.*

4 Proof Ideas of Theorem 1 ($n = 2$)

We focus on the case of $n = 2$ in this section. The proof of Theorem 1 for this case demonstrates the key ideas for the general cases, which are visualizable when $I = 3$ and $n = 2$.[15] Precisely, when we have three agents, the set of type profiles $\mathbf{T}$ is a subset of $\mathbb{R}^3$. Thus, we can illustrate each step of the proof in a 3-dimensional diagram. Technical details omitted in this section can be found in the Online Appendix C. We refer the readers to Chua et al. (2019) for the case of $n > 2$, where the proof is more tedious but the ideas remain the same.

We provide a roadmap, illustrated in Figure 1, before proceeding with more details. Mathematically, the principal's problem (1)-(5) is an infinite dimensional linear programming with infinitely many constraints. Our basic idea is to reduce the dimensions. We first identify a set of binding constraints such that the checking policy $\mathbf{q}$ is pinned down by the allocation rule $\mathbf{p}$, which means that the choice variable can be simplified from $(\mathbf{p}, \mathbf{q})$ to $\mathbf{p}$; see Section 4.1. Second, we classify the set of feasible mechanisms into convex and compact subsets; see Section 4.2. Third, within each of these convex and compact subsets, we reduce the infinite dimensional problem to a finite dimensional problem. The link between them is what we called *threshold mechanisms*, in which an I dimensional vector largely pins down the entire infinite dimensional mechanism $\mathbf{p}$; see Section 4.3. This reduction of dimensions is justified by showing that it is sufficient to restrict attentions to the threshold mechanisms; see Section 4.4. Finally, we examine the extreme points of these convex and compact subsets, and find the optimal ones among the threshold mechanisms; see Section 4.5.[16]

4.1 Simplification of the Principal's Problem

In this subsection, we simplify the principal's problem (1)-(5). The goal is to eliminate $\mathbf{q}$ from the choice variable such that the principal only needs to choose $\mathbf{p}$. We follow Ben-Porath et al. (2014) to define $\varphi_i = \inf_{t_i' \in T_i} \mathbf{E}_{\mathbf{t}_{-i}} [p_i(t_i', \mathbf{t}_{-i})]$, i.e. φ_i is the infimum of agent i's expected total assignment probability. Then by the same argument as in (Ben-Porath et al., 2014, pp. 3793-4), the principal's

[15] The techniques to analyze the continuous-type model is significantly different from that for the discrete-type model. According to the technical details of Ben-Porath et al. (2019) (the definition of $\tilde{v}_i(t_i)$ at p.565), it is not obvious how their approach can be used to analyze a continuous-type model. One may conjecture that some limiting argument will work. A seemingly possible route may be as follows. We first approximate/discretize the continuum type space by a finite type space, say, with a parameter measuring the size of a grid. Once we find the optimal allocation rule in the discrete-type model, taking limit of the parameter will lead us to an optimal allocation rule for the continuum-type model. However, taking limit requires that the solution to the discretized optimization problem is continuous in the parameter. To the best of our knowledge, we do *not* have a reliable mathematical result, e.g. something like the Berge Maximum Theorem (Aliprantis and Border, 2006, p. 570), to justify this limiting argument.

Nevertheless, once we characterize the optimal mechanisms as the n-ASMs, we do *prove* an approximation result indirectly: discretizing the continuous distributions and using the results of Ben-Porath et al. (2019) will indeed lead us to an approximately optimal mechanism for the continuous-type allocation problem. That is why we view our characterization also as a technical contribution and a nontrivial extension of Ben-Porath et al. (2019).

[16] We largely follow Ben-Porath et al. (2014) in Section 4.1. Our classification of feasible mechanisms and construction of threshold mechanisms are both conceptually novel, compared with those in Ben-Porath et al. (2014). The proof techniques in Sections 4.4 and 4.5 are substantial extensions of those in Lipman (2015) and Ben-Porath et al. (2014).

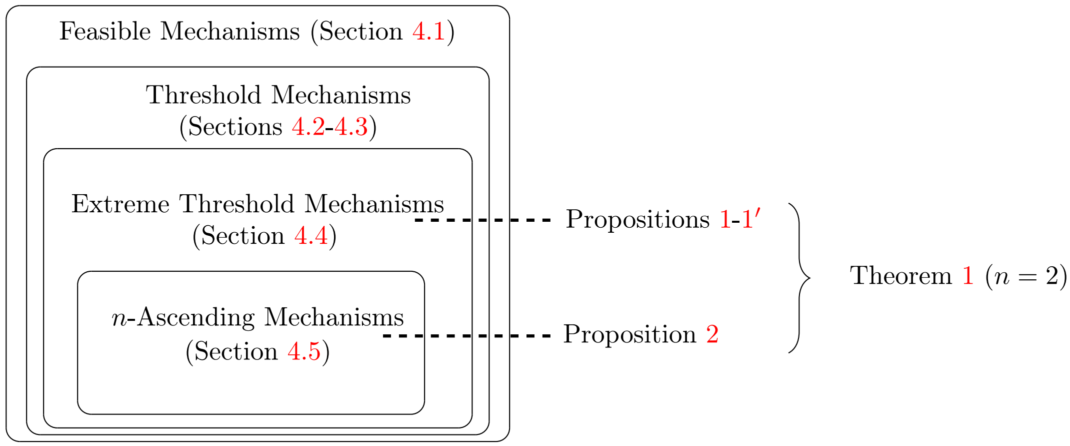


Figure 1: Proof Sketch of Theorem 1.

problem can be simplified to

$$\max_{\mathbf{p}} \quad \mathbf{E}\left(\sum_{i\in\mathcal{I}} p_i(\mathbf{t})(t_i - c_i) + \sum_{i\in\mathcal{I}} \varphi_i c_i\right) \tag{8}$$

$$\text{subject to} \quad p_i(\mathbf{t}) \in [0,1], \quad \mathbf{t}\in\mathbf{T}, \quad \forall i \in \mathcal{I},$$

$$\sum_{i\in\mathcal{I}} p_i(\mathbf{t}) \le 2, \quad \forall \mathbf{t}\in\mathbf{T}, \quad \text{and}$$

$$\varphi_i = \inf_{t_i'\in T_i} \mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i', \mathbf{t}_{-i})\right], \quad \forall i \in \mathcal{I}, \tag{9}$$

which is independent of $\mathbf{q}$. From now on, we also refer to a mechanism as $\mathbf{p}$.

We say a mechanism $\mathbf{p}$ is *feasible* if $p_i(\mathbf{t}) \in [0,1]$ for all $\mathbf{t}$ and all i, and $\sum_{i\in\mathcal{I}} p_i(\mathbf{t}) \le 2$ for all $\mathbf{t}$. Let P denote the set of feasible mechanisms, i.e.

$$P := \left\{\mathbf{p}: \quad p_i(\mathbf{t}) \in [0,1], \forall \mathbf{t}\in\mathbf{T}, \forall i \in \mathcal{I} \quad \text{and} \quad \sum_{i\in\mathcal{I}} p_i(\mathbf{t}) \le 2, \forall \mathbf{t}\in\mathbf{T}\right\}.$$

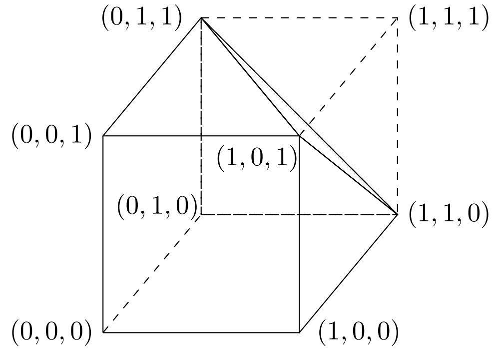
A vector $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_I) \in \mathbb{R}_+^I$ is said to be *feasible* if there exists a feasible mechanism $\mathbf{p}$ such that (9) holds. The following lemma characterizes the set of feasible $\boldsymbol{\varphi}$'s.

Lemma 2. *$\boldsymbol{\varphi}$ is feasible if and only if $\varphi_i \in [0,1]$ for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} \varphi_i \le 2$.*

Let Φ denote the set of feasible $\boldsymbol{\varphi}$'s, i.e.

$$\Phi := \left\{\boldsymbol{\varphi}: \quad \varphi_i \in [0,1], \forall i \in \mathcal{I} \quad \text{and} \quad \sum_{i\in\mathcal{I}} \varphi_i \le 2\right\}.$$

For example, the figure below illustrates Φ for the allocation problem of two goods to three agents. Two remarks are in order. First, although Φ and $\mathbf{T}$ are both subsets in $\mathbb{R}^I$, $\boldsymbol{\varphi} \in \Phi$ is interpreted as a probability profile whereas $\mathbf{t} \in \mathbf{T}$ is a type profile. Second, there may be multiple feasible mechanisms that correspond to the same vector $\boldsymbol{\varphi}$. In what follows, we solve the principal's problem by first finding the optimal $\mathbf{p}$ for a given (subset of) vector $\boldsymbol{\varphi}$ and then solving for the overall optimal $\mathbf{p}$.



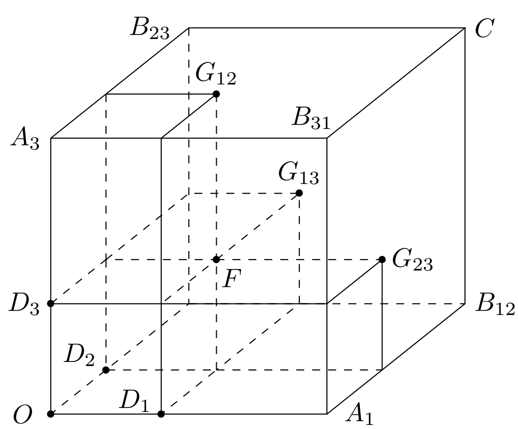
4.2 Classification of Φ and P

We classify Φ into $I+1$ subsets and denote by Φ_j, $j \in \mathcal{I} \cup \{\emptyset\}$, the j-th class. A short intuition for $\boldsymbol{\varphi} \in \Phi_j$ is that the j-th entry φ_j is "relatively larger" than φ_i, for all $i \neq j$. Similarly, for $\boldsymbol{\varphi} \in \Phi_\emptyset$, φ_i's are "relatively even." Each class will be further divided into slices which are convex and compact subsets of Φ.

We first introduce some notations for different subsets of $\mathbf{T}$. Let $\underline{v} = \min_{i\in\mathcal{I}}\{\underline{t}_i - c_i\}$ and $\bar{v} = \min_{i\in\mathcal{I}}\{\bar{t}_i - c_i\}$. For each $v \in [\underline{v}, \bar{v}]$ and each $j \in \mathcal{I} \cup \{\emptyset\}$, we define $K^j(v) \subseteq \mathbf{T}$ as the subset of type profiles such that agent j (nobody when $j = \emptyset$) has a net value which is higher than v and all other agents have net values lower than v, i.e.,

$$K^j(v) := \{\mathbf{t} \in \mathbf{T} : \quad t_j - c_j \geq v \geq t_l - c_l, \quad \forall l \neq j\}.$$

We call $K^\emptyset(v)$ the *palm*, $K^j(v)$ the j-th *finger* to facilitate our verbal description. Define the *claw* as $K(v) := K^\emptyset(v) \bigcup \left[\bigcup_{j\in\mathcal{I}} K^j(v)\right]$. The figure below illustrates $\mathbf{T}$, $K^\emptyset(v)$ and $K^j(v)$ for the allocation problem of two goods to three agents. In such an example, we assume $\underline{t}_1 - c_1 = \underline{t}_2 - c_2 = \underline{t}_3 - c_3 = \underline{v}$ and $\bar{t}_1 - c_1 = \bar{t}_2 - c_2 = \bar{t}_3 - c_3 = \bar{v}$. The point O stands for the type profile $(\underline{t}_1 - c_1, \underline{t}_2 - c_2, \underline{t}_3 - c_3)$, C for $(\bar{t}_1 - c_1, \bar{t}_2 - c_2, \bar{t}_3 - c_3)$, and F for (v, v, v). $K^\emptyset(v)$ is the cube OF and $K^j(v)$ is the pillar $D_j G_{-j}$, where $j = 1, 2, 3$.



Now we define the classes Φ_j, where $j \in \mathcal{I} \cup \{\emptyset\}$. The class $\Phi_\emptyset$ consists of slices indexed by $v \in [\underline{v}, \bar{v}]$, i.e. $\Phi_\emptyset = \bigcup_{v\in[\underline{v},\bar{v}]} \Phi_\emptyset(v)$.[17] Consider two cases. For each $v \in (\underline{v}, \bar{v}]$,

$$\Phi_\emptyset(v) := \left\{ \boldsymbol{\varphi} \in \Phi : \begin{array}{rl} \sum_{i\in\mathcal{I}} \varphi_i F_i(v+c_i) = \int_{K(v)} dF + \int_{K^\emptyset(v)} dF, & \text{and} \\ \varphi_i F_i(v+c_i) \le \int_{K(v)\setminus K^i(v)} dF, \quad \forall i \in \mathcal{I} & \end{array} \right\}.$$

Since each $\boldsymbol{\varphi} \in \Phi_\emptyset(v)$ satisfies a linear equation, a slice $\Phi_\emptyset(v)$ is a subset of an $(I-1)$-dimensional hyperplane in $\mathbb{R}^I$. That is why we call $\Phi_\emptyset(v)$ a slice. The detailed intuition of the equation and inequalities that define $\Phi_\emptyset(v)$ will be clear when we introduce threshold mechanisms in Section 4.3. For $v = \underline{v}$, the corner "slice" is a singleton defined as $\Phi_\emptyset(\underline{v}) := \{\mathbf{0}\}$.

The class Φ_j for each $j \in \mathcal{I}$ consists of slices indexed by a pair (v, v') such that $v \in [\underline{v}, \bar{v}]$ and $v' \in [v, \bar{v}]$. For notational convenience, we write $(v, v') \in [\underline{v}, \bar{v}] \times [v, \bar{v}]$. Then, $\Phi_j := \bigcup_{(v,v')\in[\underline{v},\bar{v}]\times[v,\bar{v}]} \Phi_j(v, v')$. Consider three cases. For each $j \in \mathcal{I}$ and each pair $(v, v') \in (\underline{v}, \bar{v}] \times [v, \bar{v}]$,

$$\Phi_j(v, v') := \left\{ \boldsymbol{\varphi} \in \Phi : \begin{array}{rl} \varphi_j = \prod_{k\in\mathcal{I}\setminus\{j\}} F_k(v'+c_k) + \sum_{i\neq j} [1 - F_i(v'+c_i)] \prod_{k\neq i,j} F_k(v'+c_k); & \text{and} \\ \sum_{i\in\mathcal{I}\setminus\{j\}} \varphi_i F_i(v+c_i) = \int_{K^\emptyset(v)\cup K^j(v)} dF, \qquad \text{if } |i \in \mathcal{I}\setminus\{j\} : \varphi_i > 0| \ge 2, & \\ \varphi_i = \prod_{k\neq i,j} F_k(v+c_k), \qquad \text{if } \varphi_i > 0 \text{ and } \varphi_k = 0, \quad \forall k \neq i, j & \end{array} \right\}.$$

Since for each $\boldsymbol{\varphi} \in \Phi_j(v, v')$, the j-th entry is fixed and the other $I-1$ entries generally satisfy a linear equation, we know that a slice $\Phi_j(v, v')$ is a subset of an $(I-2)$-dimensional hyperplane in $\mathbb{R}^I$. Again, the detailed intuition of the equation and inequalities will be clear in Section 4.3. But the close connection between $\Phi_\emptyset(v)$ and $\Phi_j(v, v')$ needs be clarified immediately. Note that when $F_j(v'+c_j) > 0$,

$$\prod_{k\in\mathcal{I}\setminus\{j\}} F_k(v'+c_k) + \sum_{i\neq j} [1 - F_i(v'+c_i)] \prod_{k\neq i,j} F_k(v'+c_k) = \frac{1}{F_j(v'+c_j)} \int_{K(v')\setminus K^j(v')} dF,$$

in which case $\varphi_j F_j(v'+c_j) = \int_{K(v')\setminus K^j(v')} dF$. When $F_i(v+c_i) > 0$,

$$\prod_{k\neq i,j} F_k(v+c_k) = \frac{1}{F_i(v+c_i)} \int_{K^\emptyset(v)\cup K^j(v)} dF,$$

in which case $\varphi_i F_i(v+c_i) = \int_{K^\emptyset(v)\cup K^j(v)} dF \le \int_{K(v)\setminus K^i(v)} dF$. Therefore, when $v' = v$, it is straightforward to verify that $\Phi_j(v, v) \subseteq \Phi_\emptyset(v)$. As a corollary, for each $j \in \mathcal{I}$, the class Φ_j has nonempty intersection with $\Phi_\emptyset$.

For each $j \in \mathcal{I}$, $v = \underline{v}$ and each $v' \in (\underline{v}, \bar{v}]$, we define the boundary slice

$$\Phi_j(\underline{v}, v') := \left\{ \boldsymbol{\varphi} \in \Phi : \begin{array}{rl} \varphi_j = \prod_{k\in\mathcal{I}\setminus\{j\}} F_k(v'+c_k) + \sum_{i\neq j} [1 - F_i(v'+c_i)] \prod_{k\neq i,j} F_k(v'+c_k); & \text{and} \\ \varphi_i = 0, \quad \forall i \neq j & \end{array} \right\}.$$

We define $\Phi_j(v, v')$ and $\Phi_j(\underline{v}, v')$ separately to avoid significant overlap between slices. Finally, for $v = v' = \underline{v}$, the corner "slice" is a singleton defined as $\Phi_j(\underline{v}, \underline{v}) := \{\mathbf{0}\}$.

Those slices are compact and convex by definition. The following lemma says that any $\boldsymbol{\varphi} \in \Phi$ must belong to one of the slices in Φ_j for some j, i.e. $\Phi = \left[\bigcup_{v\in[\underline{v},\bar{v}]} \Phi_\emptyset(v)\right] \cup \left[\bigcup_{j\in\mathcal{I},(v,v')\in[\underline{v},\bar{v}]\times[v,\bar{v}]} \Phi_j(v, v')\right]$.

[17] Each slice here is a compact subset of Φ. The same footnote applies to slices of Φ_j for all $j \in \mathcal{I}$.

Lemma 3. *For any vector $\boldsymbol{\varphi} \in \Phi$, there either exists a net value $v \in [\underline{v}, \bar{v}]$ such that $\boldsymbol{\varphi} \in \Phi_\emptyset(v)$, or exist $j \in \mathcal{I}$ and a pair of net values $(v, v') \in [\underline{v}, \bar{v}] \times [v, \bar{v}]$ such that $\boldsymbol{\varphi} \in \Phi_j(v, v')$.*

The following lemma characterizes the extreme points of $\Phi_\emptyset(v)$ and $\Phi_j(v, v')$, when not a singleton.

Lemma 4. *For each $v \in (\underline{v}, \bar{v}]$, the set of extreme points of $\Phi_\emptyset(v)$ is given by*

$$\Phi_\emptyset^{ex}(v) := \left\{ \boldsymbol{\varphi} \in \Phi : \quad \exists \text{ distinct } i, j \in \mathcal{I} \quad \text{s.t.} \quad \begin{array}{r} \varphi_k = 0, \quad \forall k \in \mathcal{I} \setminus \{i, j\}, \\ \varphi_j F_j(v + c_j) = \int_{K(v) \setminus K^j(v)} dF, \text{ and} \\ \varphi_i F_i(v + c_i) = \int_{K^\emptyset(v) \cup K^j(v)} dF. \end{array} \right\}.$$

For each $j \in \mathcal{I}$ and each pair $(v, v') \in (\underline{v}, \bar{v}] \times [v, \bar{v}]$, the set of extreme points of $\Phi_j(v, v')$ is given by

$$\Phi_j^{ex}(v, v') := \left\{ \boldsymbol{\varphi} \in \Phi : \quad \exists i \in \mathcal{I} \setminus \{j\} \quad \text{s.t.} \quad \begin{array}{r} \varphi_k = 0, \quad \forall k \in \mathcal{I} \setminus \{i, j\}, \\ \varphi_j F_j(v' + c_j) = \int_{K(v') \setminus K^j(v')} dF, \text{ and} \\ \varphi_i F_i(v + c_i) = \int_{K^\emptyset(v) \cup K^j(v)} dF. \end{array} \right\}.$$

We close this subsection by classifying P. By Lemma 2, we know that a classification of Φ induces a classification of P. To be precise, for each $j \in \mathcal{I} \cup \{\emptyset\}$, the set of mechanisms that corresponds to Φ_j is defined as follows:

$$P_j := \left\{ \mathbf{p} \in P : \quad \exists \boldsymbol{\varphi} \in \Phi_j \quad \text{s.t.} \quad \varphi_i = \inf_{t_i' \in T_i} \mathbf{E}_{\mathbf{t}_{-i}} \left[p_i(t_i', \mathbf{t}_{-i}) \right], \quad \forall i \in \mathcal{I} \right\}.$$

The slices of P_j, i.e. $P_\emptyset(v)$'s and $P_j(v, v')$'s, are defined analogously.

4.3 Threshold Mechanisms: Definitions

In this subsection, we introduce a *threshold mechanism* within each slice of each class of P. Let $\boldsymbol{\varphi}^{\mathbf{p}} \in \Phi$ be the vector that is induced from $\mathbf{p} \in P$ in the sense of (9). The advantage of threshold mechanisms is that the infinite dimensional $\mathbf{p}$ is largely pinned down by an I-dimensional vector $\boldsymbol{\varphi}^{\mathbf{p}}$.

We provide a brief intuition for the threshold mechanisms. The principal's objective function $\mathbf{E}\left(\sum_i p_i(\mathbf{t})(t_i - c_i) + \sum_i \varphi_i c_i\right)$ represents a trade-off between allocation efficiency and checking-cost saving, where $\sum_i \varphi_i c_i$ is the saved cost. When goods are allocated without checking agents' values, agents with low values may receive the goods in the presence of high-valued agents, i.e. there may be efficiency loss. The idea of threshold mechanisms is to allow for such efficiency loss at low-value profiles in the sense that either (i) every net value $t_i - c_i$ is low or (ii) the number of agents who have high net values is small. Particularly, the first case corresponds to the palm $K^\emptyset(v)$, where $t_i - c_i < v$ for all $i \in \mathcal{I}$; and the second case corresponds to the fingers $K^j(v)$, where $t_j - c_j > v$ and all other agents have low net values. Here is the reason why we seek to save checking cost and allow for efficiency loss: For value profiles in $K^\emptyset(v)$, agents' values are too low to be worth distinguishing, in which case we would allocate the two goods without checking anyone. For value profiles in $K^j(v)$, obviously, agent j deserves a good. However, the other agents' values are too low to be worth distinguishing, in which case we would allocate the remaining good to some $i \neq j$ without checking anyone.

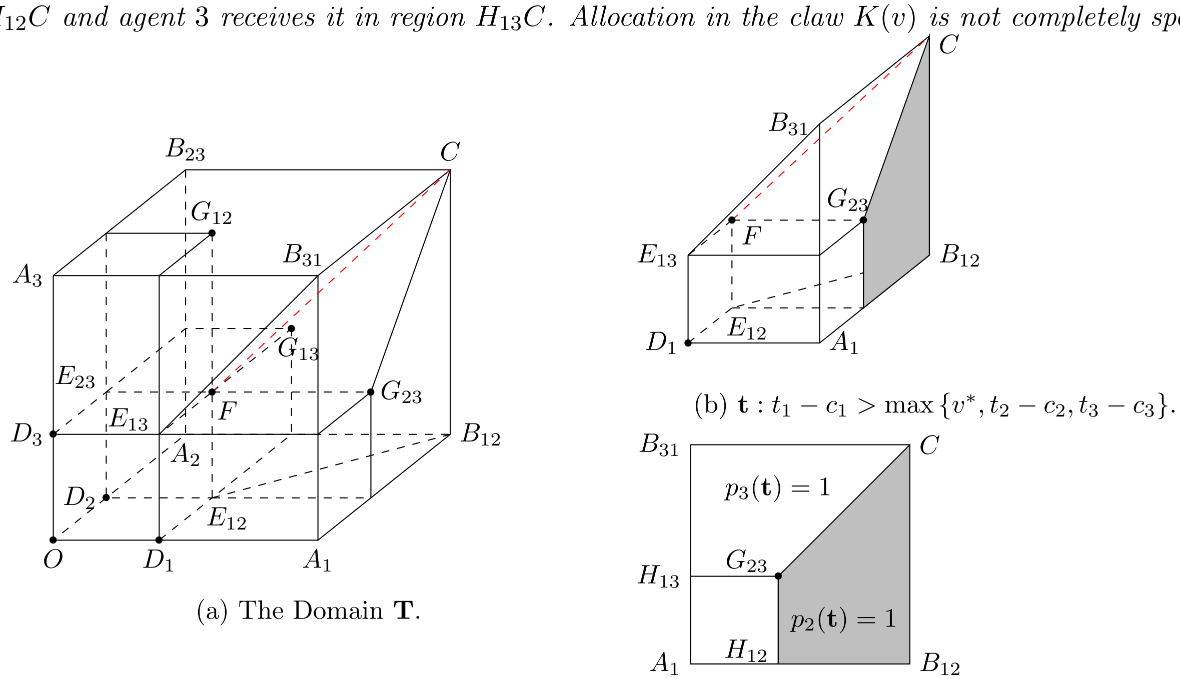
Below we first define threshold mechanisms for $j = \emptyset$, and then for $j \in \mathcal{I}$. They are (partially) illustrated in Examples 2 and 3 respectively.

Definition 3. *A feasible mechanism $\mathbf{p} \in P_\emptyset(v)$ is called a $\emptyset$-**threshold mechanism** if the following three conditions hold:*

1. *For each $\mathbf{t} \notin K(v)$, two agents with the highest net values obtain the goods, i.e. for each $i \in \mathcal{I}$,*
$$p_i(t_i, \mathbf{t}_{-i}) = \begin{cases} 1, & \text{if } |\{j \in \mathcal{I} : t_j - c_j \geq t_i - c_i\}| \leq 2; \\ 0, & \text{otherwise.}^{18} \end{cases}$$
2. *For each $i \in \mathcal{I}$ and each $\mathbf{t} \in K^i(v)$, agent i obtains one good for sure, i.e. $p_i(\mathbf{t}) = 1$.*
3. *For each $i \in \mathcal{I}$ and each t_i with $\underline{t}_i - c_i \leq t_i - c_i \leq v$, $\mathbf{E}_{\mathbf{t}_{-i}}[p_i(t_i, \mathbf{t}_{-i})] = \varphi_i^{\mathbf{p}}$.*

Condition 1 pins down the ex post allocation, condition 2 specifies the ex post allocation partially, whereas condition 3 only imposes a restriction by specifying the interim (reduced-form) allocation.

Example 2. *Consider the allocation of two goods among three agents. The entire domain $\mathbf{T}$ is included in Figure 2a for completeness. Let F be the profile of net values (v, v, v). In the subdomain indicated by Figure 2b, agent 1 receives one good for sure according to rule 1 and rule 2 (since he has the highest net value). Figure 2c further illustrates rule 1 (fixing $t_1 - c_1 = \bar{v}$), which says that the other good goes to the agent who has the second highest net value. Particularly, agent 2 receives the other good in region $H_{12}C$ and agent 3 receives it in region $H_{13}C$. Allocation in the claw $K(v)$ is not completely specified.*



(a) The Domain $\mathbf{T}$.

(b) $\mathbf{t} : t_1 - c_1 > \max\{v^*, t_2 - c_2, t_3 - c_3\}$.

(c) Allocation between agents 2 and 3.

Figure 2: A 3-Dimensional Illustration of $\mathbf{p}(\mathbf{t})$ for $\emptyset$-threshold mechanisms.

[18] We allow the mechanism to withhold some good in tied cases to avoid discussions on zero measure sets.

Definition 4. *For each $j \in \mathcal{I}$, a feasible mechanism $\mathbf{p} \in P_j(v,v')$ is called a **j-threshold mechanism** if the following three conditions hold:*

1. *For each $\mathbf{t} \notin K(v')$, two agents with the highest net values obtain the goods, i.e. for each $i \in \mathcal{I}$,*
$$p_i(\mathbf{t}) = \begin{cases} 1, & \text{if } |\{k \in \mathcal{I} : t_k - c_k \geq t_j - c_j\}| \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$
2. *Consider three cases within $K(v')$. For each $i \in \mathcal{I} \setminus \{j\}$ and each $\mathbf{t} \in K^i(v')$, agent i and agent j each obtains a good for sure, i.e. $p_i(\mathbf{t}) = p_j(\mathbf{t}) = 1$. For each $\mathbf{t} \in \left[K^{\emptyset}(v') \cup K^j(v')\right] \setminus \left[K^{\emptyset}(v) \cup K^j(v)\right]$, agent j receives one good, and the agent who has the highest net value among agents in $\mathcal{I} \setminus \{j\}$ receives the remaining good, i.e. $p_j(\mathbf{t}) = 1$ and for each $i \in \mathcal{I} \setminus \{j\}$,*
$$p_i(\mathbf{t}) = \begin{cases} 1, & \text{if } t_i - c_i > t_k - c_k \text{ for all } k \neq i, j; \\ 0, & \text{otherwise.} \end{cases}$$
Finally, for each $\mathbf{t} \in \left[K^{\emptyset}(v) \cup K^j(v)\right]$, agent j receives one good, i.e. $p_j(\mathbf{t}) = 1$.
3. *For each $i \in \mathcal{I} \setminus \{j\}$ and each t_i with $\underline{t}_i - c_i \leq t_i - c_i \leq v$, we have $\mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i, \mathbf{t}_{-i})\right] = \varphi_i^{\mathbf{p}}$.*

Conditions 1-2 pin down the ex post allocation except for $\mathbf{t} \in \left[K^{\emptyset}(v) \cup K^j(v)\right]$, whereas condition 3 only specifies the interim rule. Notably, for each t_j with $\underline{t}_j - c_j \leq t_j - c_j \leq v'$, conditions 1-2 imply that $\mathbf{E}_{\mathbf{t}_{-j}}\left[p_j(t_j, \mathbf{t}_{-j})\right] = \prod_{k \in \mathcal{I}\setminus\{j\}} F_k(v' + c_k) + \sum_{i \neq j} [1 - F_i(v' + c_i)] \prod_{k \neq i,j} F_k(v' + c_k)$, which is consistent with rule 3 in format, i.e. $\mathbf{E}_{\mathbf{t}_{-j}}\left[p_j(t_j, \mathbf{t}_{-j})\right] = \varphi_j^{\mathbf{p}}$, since $\mathbf{p} \in P_j(v,v')$.

Example 3. *Consider the allocation of two goods among three agents. Let F be the net-value profile (v,v,v) and H be the net-value profile (v',v',v'). Let $j = 1$. Two key features, in addition to those already exist in Definition 3, are as follows: First, for every $\mathbf{t} \notin K^{\emptyset}(v) \cup K^j(v)$, the ex post allocation is completely specified. Particularly, for each $\mathbf{t} \in \left[K^{\emptyset}(v') \cup K^j(v')\right] \setminus \left[K^{\emptyset}(v) \cup K^j(v)\right]$, i.e. the subdomain in the right panel of Figure 3, the remaining good goes to the agent with the highest net value among agents in $\mathcal{I} \setminus \{j\}$. Second, allocation in subdomain $K^{\emptyset}(v) \cup K^j(v)$ is not completely specified.*

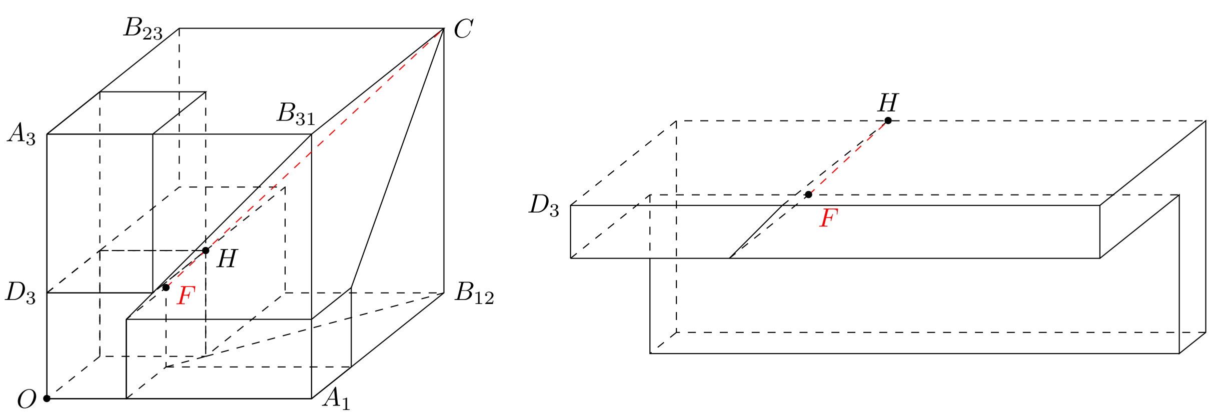


Figure 3: A 3-Dimensional Illustration of $\mathbf{p}(\mathbf{t})$ for j-Threshold Mechanisms.

The following lemma guarantees that for every feasible mechanism $\mathbf{p} \in P$, there exists a corresponding threshold mechanism.

Lemma 5. *For each $v \in [\underline{v}, \bar{v}]$ and each $\mathbf{p} \in P_\emptyset(v)$, there exists a $\emptyset$-threshold mechanism $\mathbf{p}'$ in $P_\emptyset(v)$ such that $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$. For each $j \in \mathcal{I}$, each $(v, v') \in [\underline{v}, \bar{v}] \times [v, \bar{v}]$ and each $\mathbf{p} \in P_j(v, v')$, there exists a j-threshold mechanism $\mathbf{p}'$ in $P_j(v, v')$ such that $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$.*

4.4 Threshold Mechanisms: Necessity

In this subsection, we show that without loss of generality, we can restrict our attention to threshold mechanisms, particularly the *extreme* threshold mechanisms. We say a threshold mechanism $\mathbf{p}$ is *extreme* if the induced $\boldsymbol{\varphi}^{\mathbf{p}}$ is an extreme point either in $\Phi_\emptyset(v)$ for some v or in $\Phi_j(v, v')$ for some (v, v').

Definition 5. *For each $v \in [\underline{v}, \bar{v}]$, an $\emptyset$-threshold mechanism $\mathbf{p} \in P_\emptyset(v)$ is an **extreme $\emptyset$-threshold mechanism** if $\boldsymbol{\varphi}^{\mathbf{p}} \in \Phi_\emptyset^{ex}(v)$. For each $j \in \mathcal{I}$ and each pair of $(v, v') \in [\underline{v}, \bar{v}] \times [v, \bar{v}]$, a j-threshold mechanism $\mathbf{p} \in P_j(v, v')$ is an **extreme j-threshold mechanism** if $\boldsymbol{\varphi}^{\mathbf{p}} \in \Phi_j^{ex}(v, v')$.*

An optimal mechanism is necessarily a randomization over extreme threshold mechanisms[19]:

Proposition 1. *A mechanism is optimal only if it is essentially a randomization over the optimal extreme threshold mechanisms.*

A key step to prove the proposition is to show that any optimal mechanism is equivalent to its corresponding threshold mechanism as specified in Lemma 5.

By Lemma 4, we know that any extreme $\emptyset$-threshold mechanism $\mathbf{p} \in P_\emptyset(v)$ must also be an extreme j-threshold mechanism with thresholds (v, v) for some $j \in \mathcal{I}$. Then, the following proposition is an immediate corollary of Proposition 1.

Proposition 1′. *A mechanism is optimal only if it is essentially a randomization over the optimal extreme threshold mechanisms within $P \setminus P_\emptyset$.*

4.5 Optimal (Extreme Threshold) Mechanisms

In this subsection, we solve for the optimal extreme threshold mechanisms within $P \setminus P_\emptyset$. It is clear from the proof of Lemma 5 that when $\boldsymbol{\varphi}$ is extreme, the threshold mechanism $\mathbf{p}$ such that $\boldsymbol{\varphi}^{\mathbf{p}} = \boldsymbol{\varphi}$ is unique. Therefore, there is a bijection between extreme threshold mechanisms in $P \setminus P_\emptyset$ and extreme $\boldsymbol{\varphi}$'s in $\Phi \setminus \Phi_\emptyset$. Hence, the principal's problem is simply to choose optimal extreme $\boldsymbol{\varphi}$'s in $\Phi \setminus \Phi_\emptyset$. By Lemma 4, choosing an extreme $\boldsymbol{\varphi}$ is equivalent to choosing two distinct agents i and j, and respectively two thresholds v and v' such that $v \leq v'$. Therefore, the principal's problem can be much simplified.

[19] See Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2017) for reduced ($n = 1$) or similar threshold mechanisms.

For notational convenience, we let $(t-c)|_{\mathcal{I}}^{(1)}$ be the highest net value reported by agents in $\mathcal{I}$, let $(t-c)|_{\mathcal{I}}^{(2)}$ be the second highest net value reported by agents in $\mathcal{I}$, and let $(t-c)|_{\mathcal{I}\setminus\{j\}}^{(1)}$ be the highest net value reported by agents in $\mathcal{I}\setminus\{j\}$. The principal's problem is written as follows.

$$\begin{aligned}
\max_{i,j,v,v':i\neq j,v\leq v'} \quad & \varphi_i c_i + \varphi_j c_j + \int_{\mathbf{T}\setminus K(v')} (t-c)|_{\mathcal{I}}^{(1)} + (t-c)|_{\mathcal{I}}^{(2)} dF \\
& + \int_{K(v')\setminus\left[K^{\emptyset}(v')\cup K^j(v')\right]} (t_j - c_j) + (t-c)|_{\mathcal{I}}^{(1)} dF \\
& + \int_{\left[K^{\emptyset}(v')\cup K^j(v')\right]\setminus\left[K^{\emptyset}(v)\cup K^j(v)\right]} (t_j - c_j) + (t-c)|_{\mathcal{I}\setminus\{j\}}^{(1)} dF \\
& + \int_{K^{\emptyset}(v)\cup K^j(v)} (t_j - c_j) + (t_i - c_i) dF \qquad (10)\\
\text{s.t.} \quad & \varphi_j F_j(v'+c_j) = \int_{K(v')\setminus K^j(v')} dF \quad \text{and} \qquad (11)\\
& \varphi_i F_i(v+c_i) = \int_{K^{\emptyset}(v)\cup K^j(v)} dF. \qquad (12)
\end{aligned}$$

The objective function (10) follows from Definition 4, and the constraints (11) and (12) indicate that we are indeed choosing an extreme $\boldsymbol{\varphi}$ in $\Phi\setminus\Phi_{\emptyset}$.

Let $T_{\mathcal{I}}^{j}$ be the set of type profiles such that the net value of agent j is the largest among agents in $\mathcal{I}$, i.e. $T_{\mathcal{I}}^{j} := \{\mathbf{t}\in\mathbf{T} : t_j - c_j \geq t_k - c_k, \forall k\in\mathcal{I}\setminus\{j\}\}$. The principal's objective can be rearranged into the following one:

$$\begin{aligned}
\max_{i,j,v,v':i\neq j,v\leq v'} \quad & \varphi_j c_j + \int_{\mathbf{T}\setminus\left[K(v')\cup T_{\mathcal{I}}^{j}\right]} (t-c)|_{\mathcal{I}}^{(2)} dF + \int_{K(v')\cup T_{\mathcal{I}}^{j}} (t_j - c_j) dF \\
& + \varphi_i c_i + \int_{\mathbf{T}\setminus\left[K^{\emptyset}(v)\cup K^j(v)\right]} (t-c)|_{\mathcal{I}\setminus\{j\}}^{(1)} dF + \int_{K^{\emptyset}(v)\cup K^j(v)} (t_i - c_i) dF.
\end{aligned}$$

The choice of v and i is actually a subproblem given v' and j, and vice versa; the only constraints are $i\neq j$ and $v\leq v'$. In what follows, we first drop the constraint $v\leq v'$ and analyze the two relaxed subproblems separately. Then we aggregate the solutions for the relaxed subproblems to obtain the solution for the original problem (10)-(12).

(Relaxed) Optimal v and i. We first solve for the optimal v and i by dropping $v\leq v'$.

$$\max_{i,v:i\neq j} \quad \varphi_i c_i + \int_{\mathbf{T}\setminus\left[K^{\emptyset}(v)\cup K^j(v)\right]} (t-c)|_{\mathcal{I}\setminus\{j\}}^{(1)} dF + \int_{K^{\emptyset}(v)\cup K^j(v)} (t_i - c_i) dF \quad \text{s.t.} \quad (12). \qquad (13)$$

It is straightforward to see that the objective function is independent of the type of agent j. Thus, the subproblem is a single-good allocation problem with costly verification, where the set of agents is $\mathcal{I}\setminus\{j\}$ and the allocation rule is as follows: agent i obtains the good if $\mathbf{t}\in K^{\emptyset}(v)\cup K^j(v)$; otherwise, the agent who has the highest net value obtains the good. As for the checking policy, the saved checking cost $\varphi_i c_i$ and $\varphi_i = \prod_{k\neq i,j} F_k(v+c_k)$[20] amount to saying that for each $\mathbf{t}$ such that $t_k - c_k \leq v$ for all $k\neq i,j$, no one is checked. The rest of the objective function says that for all other type profiles, the

[20]Since $\int_{K^{\emptyset}(v)\cup K^j(v)} dF = \prod_{k\neq j} F_k(v+c_k)$, we know that $\varphi_i = \prod_{k\neq i,j} F_k(v+c_k)$ when $F_i(v+c_i) > 0$.

agent who gets the good is checked. By Theorem 3 of Ben-Porath et al. (2014), we know that the subproblem has the following solution.

Lemma 6. *The relaxed problem (13) has the following solution:*

1. *When $j = 1$, the optimal $i = 2$ and the optimal $v = v_2^*$.*
2. *When $j \neq 1$, the optimal $i = 1$ and the optimal $v = v_1^*$.*

(Relaxed) Optimal v' and j. Now we investigate the optimal v' and j by dropping $v \leq v'$.

$$\max_{j,v':j\neq i} \quad \varphi_j c_j + \int_{\mathbf{T}\setminus\left[K(v')\cup T_{\mathcal{I}}^j\right]} (t-c)|_{\mathcal{I}}^{(2)} dF + \int_{K(v')\cup T_{\mathcal{I}}^j} (t_j - c_j) dF \quad \text{s.t.} \quad (11). \tag{14}$$

This is again a single-good allocation problem with costly verification, where the allocation rule is as follows: within the claw $K(v')$ and the region where agent j's net value is above v' and is the highest among all agents, the good is allocated to agent j; otherwise, the good is allocated to the agent who has the *second highest* net value. As for the checking policy, the saved checking cost $\varphi_j c_j$ and constraint (11) amount to saying the following: for all type profiles in $K(v') \setminus K^j(v')$, such that at most one agent has a net value that is above v', i.e. $|\{k \neq j : t_k - c_k > v'\}| \leq 1$, no one is checked. The rest of the objective function says that for all other type profiles, the agent who gets the good is checked.

This observation facilitates the comparison of the principal's payoffs under different choices of v''s, which in turn gives us a unique optimal v' for each j, as claimed in the following lemma.

Lemma 7. *For any fixed j in the relaxed problem (14), the unique optimal choice of v' is v_j^*.*

It is worth emphasizing that v_j^* is defined in an equivalent way as in Ben-Porath et al. (2014). Nevertheless, such a threshold has a richer interpretation than illustrated in their paper; see equation (6) and the interpretation of v_j^* in Section 3.1.

(Relaxed) Small Index First. Collect the choice variables of (10)-(12) in a tuple $(j, v'; i, v)$. We proceed to compare the principal's payoffs under two tuples $(1, v_1^*; j, v_j^*)$ and $(j, v_j^*; 1, v_1^*)$ when $j \neq 1$. The former represents an extreme 1-threshold mechanism. Although the latter may not satisfy the constraint $v' \geq v$, we still refer to it as a j-threshold mechanism for convenience and this does not affect our argument. The former mechanism delivers a higher payoff to the principal than the latter.

Lemma 8. *For any agent $j \neq 1$, the 1-threshold mechanism $(1, v_1^*; j, v_j^*)$ delivers a weakly higher payoff to the principal than the j-threshold mechanism $(j, v_j^*; 1, v_1^*)$, where the comparison is strict if $v_1^* > v_j^*$.*

Optimality and Uniqueness. The following proposition states the optimality of the extreme 1-threshold mechanism $(1, v_1^*; 2, v_2^*)$, with which we can prove Theorem 1 for the case of $n = 2$.[21]

[21] The proof for the general case, i.e. $n \geq 2$, can be found in Chua et al. (2019). Although the general proof is technically more complicated, the key ideas remain the same.

Proposition 2. *The extreme 1-threshold mechanism $(1, v_1^*; 2, v_2^*)$ is an optimal choice of $(j, v'; i, v)$ for (10)-(12). Moreover, it is the unique optimal mechanism up to equivalence-relabelling-randomization.*

Proof. Suppose $j \neq 1$. By Lemma 6, we know that $i = 1$ and $v = v_1^*$ in subproblem (13). By Lemma 7, we know that $v' = v_j^*$ in subproblem (14). Thus, the solution for the relaxed problem is $(j, v_j^*; 1, v_1^*)$ for some $j \neq 1$, which must deliver a weakly higher payoff than the optimal mechanism in the original problem. Suppose $v_1^* > v_j^*$. Lemma 8 tells us that in the relaxed problem, $(1, v_1^*; j, v_j^*)$ performs strictly better than $(j, v_j^*; 1, v_1^*)$. Then, $(1, v_1^*; j, v_j^*)$ performs strictly better than the solution to the original problem. Since $(1, v_1^*; j, v_j^*)$ is a feasible mechanism satisfying $v' > v$, we have arrived at a contradiction. Therefore, we must have either $j = 1$ or $j \neq 1$ but $v_j^* = v_1^*$, where in the latter case we can relabel the agents to have $j = 1$. Thus, without loss of generality, we have $j = 1$. Again, by Lemma 6, we know that $i = 2$ with $v = v_2^*$. By Lemma 7, for $j = 1$, the optimal threshold is v_1^*. Hence, $(1, v_1^*; 2, v_2^*)$ is an optimal mechanism for the relaxed problem. Since $(1, v_1^*; 2, v_2^*)$ satisfies $v' \geq v$, it is optimal for the original problem. $\square$

Proof of Theorem 1 for $n = 2$. We prove the sufficiency part first. Let $(\mathbf{p}^{ASM}, \mathbf{q}^{ASM})$ denote the 2-ASM in Definition 1. Let $\mathbf{p}$ denote the allocation rule of $(1, v_1^*; 2, v_2^*)$. By comparing Definition 1 and Definition 4, it is straightforward to verify that $\mathbf{p}$ is equivalent to $\mathbf{p}^{ASM}$.[22] Next, we retrieve the checking policy of the mechanism $(1, v_1^*; 2, v_2^*)$, denoted by $\mathbf{q}$, by requiring that $\mathbf{E}_{\mathbf{t}_{-i}}[q_i(t_i, \mathbf{t}_{-i})] = \mathbf{E}_{\mathbf{t}_{-i}}[p_i(t_i, \mathbf{t}_{-i})] - \varphi_i^{\mathbf{p}}$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Again, it is straightforward to verify that any eligible $\mathbf{q}$ that satisfies the above requirement must be equivalent to $\mathbf{q}^{ASM}$.[23] Thus, $(\mathbf{p}, \mathbf{q})$ is equivalent to the 2-ASM. Since the extreme 1-threshold mechanism $(1, v_1^*; 2, v_2^*)$, or $(\mathbf{p}, \mathbf{q})$, is optimal, we know that the 2-ASM is optimal. As a result, essential randomizations of 2-ASMs are also optimal.

Now suppose a mechanism is optimal. By Proposition 1 or 1′, it is essentially one (or a randomization) of the extreme threshold mechanisms. Since $(1, v_1^*; 2, v_2^*)$, or the 2-ASM, is the *unique* optimal extreme threshold mechanism up to equivalence-relabelling-randomization (Proposition 2), we know that the optimal mechanism must be essentially one (or a randomization) of the 2-ASMs. $\square$

5 Concluding Remarks

In this paper, we characterize the mechanisms which maximize the net expected value to the principal from allocating multiple identical goods. Such an optimal mechanism specifies a cut-off index for each type profile, which divides the agents into two groups. In the first group, the principal allocates goods without learning agents' types, which incurs no checking cost but may result in allocation inefficiency.

[22] To have a more transparent comparison, one can specify the ex post allocation $\mathbf{p}(\mathbf{t})$ for all $\mathbf{t} \in \mathbf{T}$ according to Definition 4. This is also done in the proof of Lemma 5.

[23] For example, the checking policies after subproblem (13) and subproblem (14) constitute such an eligible $\mathbf{q}$.

In the second group, the remaining goods are allocated efficiently to the agents who have the highest net values, which requires these agents being checked and leads to some checking cost. Our optimal mechanism balances the benefit of efficient allocation and the cost of checking agents. The optimal mechanism reduces to the optimal favored-agent mechanism in Ben-Porath et al. (2014) when we have only one good, and provides an extension of Ben-Porath et al. (2019) when their approach is applied to allocation problems with costly verification. Moreover, the optimal mechanism has an obviously strategy-proof implementation.

Three of our assumptions are of further interest. First, the goods are identical, which substantially simplifies the incentive compatibility constraints. Particularly, the constraints are formulated merely by the probability of obtaining a good and the probability of being checked, which only depend on agents' reported types instead of their true types. Second, agents' types are independently distributed. This assumption echoes the use of agent-specific thresholds, and it allows for interpretations (properties) of the thresholds that hold regardless of the other agents' types. Third, the principal learns an agent's type once she inspects the agent. This allows the principal to punish an agent who lies in the most efficient way: once found lying, the agent certainly gets nothing. Again, it significantly simplifies the problem. Relaxing any of those assumptions may lead to important future research directions.

Appendix A Proof of Lemma 1

The "only-if" part. Suppose $p_i(\mathbf{t}) = 1$. According to the Ascending Algorithm, there are two possibilities for agent i to obtain a good. First, $i \le n$ and he is not challenged. Then within the group $\{i+1, \dots, I\}$, only agents who are removed with a good may have a net value that is larger than v_i^*. Therefore, $|\{j \in \{i+1, \dots, I\} : t_j - c_j > v_i^*\}| \le n - i$. Since $v_j^* \le v_i^*$ for all $j \in \{i+1, \dots, I\}$, we have

$$|\{j \in \{i+1, \dots, I\} : w_j > w_i\}| \le n - i. \tag{15}$$

Obviously,

$$|\{j \in \{1, \dots, i-1\} : w_j \ge w_i\}| \le i - 1. \tag{16}$$

Note that

$$\begin{aligned}&|\{j \in \mathcal{I} : w_j > w_i\}| + |\{j \in \{1, \dots, i-1\} : w_j = w_i\}| \\ =&|\{j \in \{i+1, \dots, I\} : w_j > w_i\}| + |\{j \in \{1, \dots, i-1\} : w_j \ge w_i\}|.\end{aligned} \tag{17}$$

Hence, the desired inequality (7) is derived by summing up (15) and (16).

Second, i obtains a good by successfully challenging some k with $k \le n$ and $k < i$. Then $t_i - c_i > \max\{v_k^*, t_k - c_k\}$. Moreover, within the group $\{k+1, \dots, I\}$, only agents who are removed with a good may have a net value that is larger than $t_i - c_i$ (or equal to $t_i - c_i$ in the tied case); otherwise, i

cannot be the successful challenger to k. Hence,

$$|\{j \in \{k+1, \ldots, I\} : t_j - c_j > t_i - c_i\}| + |\{j \in \{k+1, \ldots, i-1\} : t_j - c_j = t_i - c_i\}| \leq n - k.$$

Since $v_j^* \leq v_k^* < t_i - c_i$ for all $j \in \{k+1, \ldots, I\}$, we have

$$|\{j \in \{k+1, \ldots, I\} : w_j > w_i\}| + |\{j \in \{k+1, \ldots, i-1\} : w_j = w_i\}| \leq n - k. \tag{18}$$

Obviously,

$$|\{j \in \{1, \ldots, k-1\} : w_j \geq w_i\}| \leq k - 1. \tag{19}$$

Note that $t_i - c_i > w_k = \max\{v_k^*, t_k - c_k\}$ implies $w_i > w_k$. Then summing up (18) and (19), together with (17), leads to the desired inequality (7).

The "if" part. Now suppose (7) holds for agent i under $\mathbf{t}$. If $w_i = v_i^*$, i.e. $v_i^* \geq t_i - c_i$, then $i \leq n$. Obviously, we have $v_j^* \geq v_i^*$ for all $j \in \{1, \ldots, i-1\}$, which implies $w_j \geq w_i$. Therefore, $|\{j \in \{1, \ldots, i-1\} : w_j \geq w_i\}| = i-1$. By (7) and (17), we have $|\{j \in \{i+1, \ldots, I\} : w_j > w_i\}| \leq n-i$. This in turn implies $|\{j \in \{i+1, \ldots, I\} : t_j - c_j > v_i^*\}| \leq n - i$, i.e. agent i is not challenged. Hence, $p_i(\mathbf{t}) = 1$.

If $w_i = t_i - c_i$, i.e. $t_i - c_i > v_i^*$, we need to consider two cases. When $i \leq n$ and i is not successfully challenged, we are done as agent i gets a good.

The second case is that $i \leq n$ but i is successfully challenged. Since the net value of the successful challenger is decreasing as the Ascending Algorithm proceeds, we know that all successful challengers in $\{i+1, \ldots, I\}$ have net values larger than $t_i - c_i$. That is,

$$|\{j \in \{i+1, \ldots, I\} : t_j - c_j > t_i - c_i\}| \geq n - (i-1) \quad \text{or} \quad |\{j \in \{i+1, \ldots, I\} : w_j > w_i\}| \geq n - (i-1),$$

which implies by (7) and (17) that $|\{j \in \{1, \ldots, i-1\} : w_j \geq w_i\}| \leq i - 2$. Therefore, $w_i > w_j$ and thus $t_i - c_i > w_j$ for some $j \in \{1, \ldots, i-1\}$, which makes i a successful challenger. Hence, $p_i(\mathbf{t}) = 1$.

When $i > n$, by (7) and (17), we know that

$$\begin{aligned}
&|\{j \in \{1, \ldots, n\} : w_j \geq t_i - c_i\}| + |\{j \in \{n+1, \ldots, i-1\} : w_j \geq t_i - c_i\}| + |\{j \in \{i+1, \ldots, I\} : w_j > t_i - c_i\}| \\
=&|\{j \in \{1, \ldots, i-1\} : w_j \geq w_i\}| + |\{j \in \{i+1, \ldots, I\} : w_j > w_i\}| \\
\leq& n - 1,
\end{aligned}$$

which says that agent i fails in challenging favored agents for at most $n-1$ times. Hence, i successes once and $p_i(\mathbf{t}) = 1$.

Appendix B Proof of Theorem 2

Suppose agent i is called to play. Regardless of the history $h \in I_i(t_i)$, we could have $p_i = 1$ only if $i = m_k$ for some k, $q_i = 1$ and $t_i' = t_i$. Therefore, whenever $t_i'' \neq t_i$, $p_i(h, t_i'', \mathbf{t}_{-i}'', t_i) \equiv 0$ regardless of h

and $\mathbf{t}''_{-i}$, which means that

$$\sup_{h \in I_i(t_i), \mathbf{t}''_{-i} \in \mathbf{T}_{-i}} p_i(h, t''_i, \mathbf{t}''_{-i}, t_i) = \inf_{h \in I_i(t_i), \mathbf{t}''_{-i} \in \mathbf{T}_{-i}} p_i(h, t''_i, \mathbf{t}''_{-i}, t_i) = 0.$$

For the truthful strategy $t'_i = t_i$, we consider three cases. First, any agent $i > n$ with $t_i - c_i \leq v^*_n$ is indifferent between any strategies because p_i will be zero anyway.[24] Thus, $t'_i = t_i$ is trivially an obviously dominant strategy. Second, agent $i = n$ with $t_i - c_i \leq v^*_n$ is indifferent between any strategies, because his utility is purely determined by $\mathbf{t}'_{-i}$ and $\mathbf{t}_{-i}$. Again, $t'_i = t_i$ is trivially an obviously dominant strategy. Finally, in all other scenarios, there are $h \in I_i(t_i)$ and $\mathbf{t}''_{-i}$ such that $p_i(h, t_i, \mathbf{t}''_{-i}, t_i) = 1$ as well as $h \in I_i(t_i)$ and $\mathbf{t}''_{-i}$ such that $p_i(h, t_i, \mathbf{t}''_{-i}, t_i) = 0$, which can be seen from an example with two goods and three agents as follows (where $i = 1, 2, 3$ and $\mathbf{t}''_{-i} = \mathbf{t}_{-i}$):

$$v^*_2 < v^*_1 < t_3 - c_3 < t_2 - c_2 < t_1 - c_1$$

$$v^*_2 < v^*_1 < t_2 - c_2 < t_1 - c_1 < t_3 - c_3$$

$$v^*_2 < v^*_1 < t_1 - c_1 < t_3 - c_3 < t_2 - c_2.$$

Therefore, we have

$$\sup_{h \in I_i(t_i), \mathbf{t}''_{-i} \in \mathbf{T}_{-i}} p_i(h, t_i, \mathbf{t}''_{-i}, t_i) = 1 \quad \text{and} \quad \inf_{h \in I_i(t_i), \mathbf{t}''_{-i} \in \mathbf{T}_{-i}} p_i(h, t_i, \mathbf{t}''_{-i}, t_i) = 0.$$

As a result, $t'_i = t_i$ is an obviously dominant strategy for agent i but not any other strategy.

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[24] One can apply the argument in Footnote 10 of Ben-Porath et al. (2014) to make i strictly prefer truth-telling to lying, without changing the allocation and checking cost on equilibrium path. The same comment applies to the second case.

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Appendix C Technical Details for Section 4

C.1 Proof of Lemma 2.

Proof. **The "only-if" part.** Suppose $\boldsymbol{\varphi}$ is feasible, i.e. there exists a feasible mechanism $\mathbf{p}$ such that (9) holds. It is straightforward to see that $\varphi_i \in [0,1]$ since $p_i(\mathbf{t}) \in [0,1]$ for all $\mathbf{t}$. Moreover,

$$\begin{aligned}\sum_i \varphi_i =& \sum_i \inf_{t_i' \in T_i} \mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i', \mathbf{t}_{-i})\right] \\ \leq& \sum_i \int_{T_i} \mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i', \mathbf{t}_{-i})\right] dF_i(t_i') \\ =& \sum_i \mathbf{E}_t\left[p_i(\mathbf{t})\right] \\ =& \mathbf{E}_t\left[\sum_i p_i(\mathbf{t})\right] \\ \leq& 2.\end{aligned}$$

The "if" part. Suppose $\varphi_i \in [0,1]$ for all i and $\sum_i \varphi_i \leq 2$. Then the mechanism $\mathbf{p}$ satisfying $p_i(\mathbf{t}) = \varphi_i$ for all $\mathbf{t}$ and all i is a feasible mechanism that induces $\boldsymbol{\varphi}$ via (9). □

C.2 Proofs for Section 4.2

Proof of Lemma 3. We consider three mutually exclusive cases.

Case 1. $\boldsymbol{\varphi} = \mathbf{0}$**.** Obviously, we have $\boldsymbol{\varphi} \in \Phi_\emptyset(\underline{v})$ and $\boldsymbol{\varphi} \in \Phi_j(\underline{v}, \underline{v})$ for all $j \in \mathcal{I}$.

Case 2. There exists $j \in \mathcal{I}$ **such that** $\varphi_j > 0$ **and** $\varphi_i = 0$ **for all** $i \neq j$**.**

We consider v' such that:

$$\varphi_j = \prod_{k \in \mathcal{I} \setminus \{j\}} F_k(v' + c_k) + \sum_{i \neq j} \left[1 - F_i(v' + c_i)\right] \prod_{k \neq i,j} F_k(v' + c_k). \tag{20}$$

Since the right-hand-side is strictly increasing in v' once it leaves 0 and before it reaches 1, (20) has a unique solution v'. Since $\varphi_j > 0$, we know that $v' > \underline{v}$. Therefore, $\boldsymbol{\varphi} \in \Phi_j(\underline{v}, v')$.

Case 3. There exist distinct $j, k \in \mathcal{I}$ **such that** $\varphi_j > 0$ **and** $\varphi_k > 0$**.**

We consider the following equation and focus on the variable $\hat{v}$:

$$\sum_{i \in \mathcal{I}} \varphi_i F_i(\hat{v} + c_i) = \int_{K(\hat{v})} dF + \int_{K^\emptyset(\hat{v})} dF. \tag{21}$$

Obviously, $\underline{v}$ is a solution to (21), which we refer to as the trivial solution. We proceed to argue that (21) has a nontrivial solution $\hat{v} > \underline{v}$. On the one hand, at $\hat{v} = \bar{v}$, we have

$$\sum_{i \in \mathcal{I}} \varphi_i F_i(\bar{v} + c_i) = \sum_{i \in \mathcal{I}} \varphi_i \leq 2 = \int_{K(\bar{v})} dF + \int_{K^\emptyset(\bar{v})} dF.$$

On the other hand, denote the second lowest net value by $\underline{t}_s - c_s$. At $\hat{v} = \underline{t}_s - c_s$, the right derivative of the right-hand-side of (21) with respect to $\hat{v}$ is zero. In contrast, at least one of $\varphi_j F_j(\hat{v} + c_j)$ and

$\varphi_k F_k(\hat{v}+c_k)$ has a strictly positive right derivative at $\hat{v} = \underline{t}_s - c_s$. Therefore, for a small enough $\epsilon > 0$, we have

$$\sum_{i\in\mathcal{I}} \varphi_i F_i(\underline{t}_s - c_s + \epsilon + c_i) > \int_{K(\underline{t}_s - c_s+\epsilon)} dF + \int_{K^{\emptyset}(\underline{t}_s - c_s+\epsilon)} dF.$$

Since both sides of (21) are continuous in $\hat{v}$, we know that there exists a solution $\hat{v}$ for (21) such that $\underline{v} \le \underline{t}_s - c_s < \hat{v} \le \bar{v}$.

If for every $i \in \mathcal{I}$, we have

$$\varphi_i F_i(\hat{v}+c_i) \le \int_{K(\hat{v})\setminus K^i(\hat{v})} dF,$$

then $\boldsymbol{\varphi} \in \Phi_{\emptyset}(\hat{v})$.

Suppose there exists j such that

$$\varphi_j F_j(\hat{v}+c_j) > \int_{K(\hat{v})\setminus K^j(\hat{v})} dF. \tag{22}$$

We define v' implicitly as the solution to

$$\varphi_j F_j(v'+c_j) = \int_{K(v')\setminus K^j(v')} dF \tag{23}$$

in the interval $(\hat{v}, \bar{v}]$. Such a solution exists for the following reasons. First, at $v = \bar{v}$, we have

$$\varphi_j F_j(\bar{v}+c_j) = \varphi_j \le 1 = \int_{K(\bar{v})\setminus K^j(\bar{v})} dF.$$

Second, we have (22). Since both sides of (23) are continuous in $\hat{v}$,, we know that there exists a solution to (23) in $(\hat{v}, \bar{v}]$.

Consider two mutually exclusive subcases.

Subcase 3.1. There exists $i \in \mathcal{I} \setminus \{j\}$ such that $\varphi_i > 0$ and $\varphi_k = 0$ for all $k \ne i, j$. We define v implicitly as the solution to

$$\varphi_i = \prod_{k\ne i,j} F_k(v+c_k). \tag{24}$$

Arguments similar to those for the nontrivial solution to (21) would show that we have a nontrivial solution to (24), i.e. $v > \underline{v}$. If $v > \underline{t}_i - c_i$, then we know that

$$\varphi_i F_i(v+c_i) = \int_{K^{\emptyset}(v)} dF + \int_{K^j(v)} dF,$$

which implies $v < \hat{v}$ since (21) and (22) imply that

$$\varphi_i F_i(\hat{v}+c_i) < \int_{K^{\emptyset}(\hat{v})} dF + \int_{K^j(\hat{v})} dF.$$

In this case, we have $v < v'$ and thus $\boldsymbol{\varphi} \in \Phi_j(v, v')$.

If $v < \underline{t}_i - c_i$, then we know that

$$\int_{K^{\emptyset}(v)} dF + \int_{K^j(v)} dF = \prod_{k\ne j} F_k(v+c_k) = 0,$$

which implies $v < \hat{v}$ since (21) and (22) imply that

$$0 \le \varphi_i F_i(\hat{v}+c_i) < \int_{K^{\emptyset}(\hat{v})} dF + \int_{K^j(\hat{v})} dF = \prod_{k\ne j} F_k(\hat{v}+c_k).$$

In this case, again, we have $v < v'$ and $\boldsymbol{\varphi} \in \Phi_j(v, v')$.

Subcase 3.2. There exist distinct $i, k \in \mathcal{I} \setminus \{j\}$ such that $\varphi_i > 0$ and $\varphi_k > 0$. We define v as the solution to

$$\sum_{i \in \mathcal{I} \setminus \{j\}} \varphi_i F_i(v + c_i) = \int_{K^{\emptyset}(v) \cup K^j(v)} dF \tag{25}$$

in $(\underline{v}, \hat{v})$, where the existence of such a nontrivial solution follows from similar argument as for the nontrivial solution to (21). Therefore, we have $\boldsymbol{\varphi} \in \Phi_j(v, v')$. $\square$

Proof of Lemma 4. We characterize the extreme points of $\Phi_{\emptyset}(v)$ with $v \in (\underline{v}, \bar{v}]$. The characterization for the extreme points of $\Phi_j(v, v')$ is similar and much easier, which is omitted.

We first show that every $\boldsymbol{\varphi} \in \Phi^{ex}_{\emptyset}(v)$ is an extreme point. Let $\varphi_1, \varphi_2 > 0$ and $\varphi_k = 0$ for all $k \neq i, j$. Suppose to the contrary that there exist $\boldsymbol{\varphi}^1 \in \Phi_{\emptyset}(v)$ and $\boldsymbol{\varphi}^2 \in \Phi_{\emptyset}(v)$ such that $\boldsymbol{\varphi}^1 \neq \boldsymbol{\varphi}^2$ and there exists some $\lambda \in (0,1)$ such that $\lambda \boldsymbol{\varphi}^1 + (1-\lambda)\boldsymbol{\varphi}^2 = \boldsymbol{\varphi}$. Then we have $\varphi^1_k = \varphi^2_k = 0$ for all $k \neq i, j$ and that

$$\lambda \varphi^1_k + (1-\lambda)\varphi^2_k = \varphi_k, \quad k = i, j.$$

If $\varphi^1_j \neq \varphi^2_j$, then w.l.o.g.,

$$\varphi^1_j F_j(v + c_j) < \int_{K(v) \setminus K^j(v)} dF < \varphi^2_j F_j(v + c_j),$$

which is a contradiction to $\boldsymbol{\varphi}^2 \in \Phi_{\emptyset}(v)$. Thus, $\varphi^1_j = \varphi^2_j$ and

$$\varphi^1_j F_j(v + c_j) = \varphi^2_j F_j(v + c_j) = \int_{K(v) \setminus K^j(v)} dF.$$

Since $\boldsymbol{\varphi}^1 \neq \boldsymbol{\varphi}^2$, we must have $\varphi^1_i \neq \varphi^2_i$. Then w.l.o.g.,

$$\varphi^1_i F_i(v + c_i) < \int_{K^{\emptyset}(v)} dF + \int_{K^j(v)} dF < \varphi^2_i F_i(v + c_i),$$

which implies that

$$\varphi^2_j F_j(v + c_j) + \varphi^2_i F_i(v + c_i) > \int_{K^{\emptyset}(v)} dF + \int_{K(v)} dF,$$

a contradiction to $\boldsymbol{\varphi}^2 \in \Phi_{\emptyset}(v)$. Hence, $\boldsymbol{\varphi}$ must be an extreme point.

Next we show that there is no other extreme points. Fix any feasible $\boldsymbol{\varphi}' \in \Phi_{\emptyset}(v)$. We proceed to show that $\boldsymbol{\varphi}'$ is a convex combination of the points in $\Phi^{ex}_{\emptyset}(v)$. Let $\boldsymbol{\varphi}(j, i)$ denote the point such that $\varphi_k = 0$, for all $k \neq i, j$,

$$\varphi_j F_j(v + c_j) = \int_{K(v) \setminus K^j(v)} dF \text{ and } \varphi_i F_i(v + c_i) = \int_{K^{\emptyset}(v)} dF + \int_{K^j(v)} dF.$$

To simplify notations, we write F_i for $F_i(v + c_i)$ and $\int_K$ for $\int_{K(v)} dF$. We define a feasible $\boldsymbol{\varphi}^1$ as

$$\boldsymbol{\varphi}^1 := \begin{pmatrix} \frac{1}{F_1(v+c_1)} \int_{K \setminus K^1} \\ \frac{\varphi'_2}{\sum_{l \neq 1} \varphi'_l F_l} \left(\int_{K^1} + \int_{K^{\emptyset}} \right) \\ \cdots \\ \frac{\varphi'_I}{\sum_{l \neq 1} \varphi'_l F_l} \left(\int_{K^1} + \int_{K^{\emptyset}} \right) \end{pmatrix} = \begin{pmatrix} \frac{1}{F_1(v+c_1)} \int_{K \setminus K^1} \\ \frac{\left(\int_{K^1} + \int_{K^{\emptyset}} \right)}{\sum_{l \neq 1} \varphi'_l F_l} \varphi'_2 \\ \cdots \\ \frac{\left(\int_{K^1} + \int_{K^{\emptyset}} \right)}{\sum_{l \neq 1} \varphi'_l F_l} \varphi'_I \end{pmatrix}.$$

Then we have

$$\boldsymbol{\varphi}^1 = \frac{\varphi_2' F_2}{\sum_{l\neq 1}\varphi_l' F_l}\boldsymbol{\varphi}(1,2) + \cdots + \frac{\varphi_I' F_I}{\sum_{l\neq 1}\varphi_l' F_l}\boldsymbol{\varphi}(1,I),$$

i.e. $\boldsymbol{\varphi}^1$ is a convex combination of $\boldsymbol{\varphi}(i,j)$'s. Similarly, we define $\boldsymbol{\varphi}^i$ for other i's. Then, the matrix $M := (\boldsymbol{\varphi}^1,\ldots,\boldsymbol{\varphi}^I)$ has full rank unless $\boldsymbol{\varphi}'$ is identical to one of the $\boldsymbol{\varphi}^i$'s, in which case we are done. In all other cases, full rank of M implies that $M\beta = \boldsymbol{\varphi}'$ has a solution β. Moreover, $\beta_i \geq 0$ since $\varphi_i F_i(v+c_i) \leq \int_{K\setminus K^i}$, which means that $\boldsymbol{\varphi}'$ is a convex combination of the $\boldsymbol{\varphi}$'s in $\Phi_\emptyset^{ex}(v)$. □

C.3 Proofs for Section 4.3

Proof of Lemma 5. For $j=\emptyset$, we prove the lemma by two parts.

Part 1. Construction. We construct $\mathbf{p}'$ according to rule *1* of Definition 3 on the subdomain $\mathbf{T}\setminus K(v)$. On the subdomain $K(v)$, $\mathbf{p}'$ is constructed by rule *2* and the algorithm below, which "squeezes" $\sum_{i\in\mathcal{I}}\varphi_i F_i(v+c_i)$ into the "capacity-constrained claw." Intuitively, the number $\varphi_i F_i(v+c_i)$ for each agent i represents a probability mass with which agent i receives one good when her net value is no more than v. The *capacity constraints* within the claw mean that (i) for net-value profiles in the palm, two goods are available to be allocated, and (ii) for net-value profiles in each of the fingers, only one good is available. The capacity-constrained claw has a capacity-weighted probability mass $\int_{K(v)} dF + \int_{K^\emptyset(v)} dF$. Formally, the algorithm guarantees that $\sum_{i\in\mathcal{I}}\varphi_i F_i(v+c_i) = \int_{K(v)} dF + \int_{K^\emptyset(v)} dF$.

$\boldsymbol{\varphi}^{\mathbf{p}}$ Satisfying Algorithm.

Set $\boldsymbol{\phi} := \boldsymbol{\varphi}^{\mathbf{p}}$. *Set*

$$\mathbf{O} := \left(\int_{K^1(v)} dF, \ldots, \int_{K^i(v)} dF, 2\int_{K^\emptyset(v)} dF\right).$$

For each $i = 1,\ldots,I$,

Step i. *This step has up to* $I+1$ *rounds. For each round* $k = i+1,\ldots,I,1,\ldots,i-1,\emptyset$, *we consider two mutually exclusive cases.*

1. *If* $\phi_i F_i(v+c_i) > O_k > 0$, *then set* $p_i'(\mathbf{t}) := \frac{O_k}{\int_{K^k(v)} dF}$ *for all* $\mathbf{t}\in K^k(v)$.
   Set $\phi_i := \phi_i - \frac{O_k}{F_i(v+c_i)}$. Set $O_k := 0$. If $\phi_i F_i(v+c_i) > 0$, proceed to the next round. If $\phi_i F_i(v+c_i) = 0$ and $i < I$, go to Step $i+1$. If $\phi_i F_i(v+c_i) = 0$ and $i = I$, stop.
2. *If* $\phi_i F_i(v+c_i) \in (0, O_k]$, *then set* $p_i'(\mathbf{t}) = \frac{\phi_i F_i(v+c_i)}{\int_{K^k(v)} dF}$ *for all* $\mathbf{t}\in K^k(v)$.
   Set $O_k := O_k - \phi_i F_i(v+c_i)$. Set $\phi_i := 0$ and $k_i^* = k$, i.e. k_i^* is the round index where Step i is completed (which will be used in part 2). If $i < I$, go to Step $i+1$; otherwise, stop.

Obviously, $p_i'(\mathbf{t}) \in [0,1]$ for all i and all $\mathbf{t}\in K^k(v)$. Since $\mathbf{p}\in P_\emptyset(v)$, we know that $\boldsymbol{\varphi}^{\mathbf{p}} \in \Phi_\emptyset(v)$. Therefore, as long as the 1-st step goes to round $k=\emptyset$, we must have $\phi_1 F_1(v+c_1) \leq \int_{K^\emptyset(v)} dF$. Induction shows that as long as the i-th step goes to round $k=\emptyset$, we must have $\phi_i F_i(v+c_i) \leq$

$\min\left\{O_{I+1}, \int_{K^{\emptyset}(v)} dF\right\}$. Therefore, $\mathbf{p}'$ is feasible on $K(v)$. It is straightforward to verify that $\mathbf{p}'$ is feasible on $\mathbf{T} \setminus K(v)$. Thus, $\mathbf{p}'$ is a feasible mechanism.

Part 2. Verification We proceed to show rule *3* of Definition 3 and $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$ are satisfied. For each i and each $t_i - c_i \in [\underline{v}, v]$, we know by rule *1* that agent i gets (some fraction of) one good only in the region $K(v) \setminus K^i(v)$. Since $\mathbf{E}_{\mathbf{t}_{-i}}\left[p_i'(t_i, \mathbf{t}_{-i})\right]$ is independent of t_i in the algorithm, we know that

$$\begin{aligned}\mathbf{E}_{\mathbf{t}_{-i}}\left[p_i'(t_i, \mathbf{t}_{-i})\right] =& \frac{1}{F_i(v+c_i)} \sum_{k=i+1}^{k_i^*-1} \int_{K^k(v)} \frac{O_k}{\int_{K^k(v)} dF} dF + \int_{K^{k_i^*}(v)} \frac{\phi_i F_i(v+c_i)}{\int_{K^{k_i^*}(v)} dF} dF \\ =& \varphi_i^{\mathbf{p}},\end{aligned} \tag{26}$$

where the summation goes in the order $k = i+1, \ldots, I, 1, \ldots, i-1, \emptyset$. Hence, rule *3* is satisfied.

For agent i, if $t_i - c_i \in (v, \bar{v}]$, then agent i gets one good if and only if it has the highest or the second highest net value. Thus,

$$\begin{aligned}\mathbf{E}_{\mathbf{t}_{-i}}[p_i'(t_i, \mathbf{t}_{-i})] =& \prod_{k \in \mathcal{I} \setminus \{i\}} F_k(t_i - c_i + c_k) + \sum_{j \in \mathcal{I} \setminus \{i\}} [1 - F_j(t_i - c_i + c_j)] \prod_{k \in \mathcal{I} \setminus \{i,j\}} F_k(t_i - c_i + c_k) \\ \geq& \prod_{k \in \mathcal{I} \setminus \{i\}} F_k(v + c_k) + \sum_{j \in \mathcal{I} \setminus \{i\}} [1 - F_j(v + c_j)] \prod_{k \in \mathcal{I} \setminus \{i,j\}} F_k(v + c_k) \\ \geq& \varphi_i^{\mathbf{p}},\end{aligned} \tag{27}$$

where the second inequality follows from $\boldsymbol{\varphi}^{\mathbf{p}} \in \Phi_{\emptyset}(v)$ and the first inequality follows because the right-hand-side expression is increasing in $t_i - c_i$. Therefore, (26)-(27) imply that $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$, which also implies that $\mathbf{p}'$ is a $\emptyset$-threshold mechanism with threshold v. This completes the proof for $j = \emptyset$.

For $j \neq \emptyset$, we prove the case where there are at least two agents other than j who have positive φ_i's. The other cases can be proved with simpler arguments and, thus, are omitted. The proof also consists of two parts.

Part 1. Construction We define $\mathbf{p}'$ first. For every $\mathbf{t} \in K^{\emptyset}(v) \cup K^j(v)$, set

$$p_i'(\mathbf{t}) = \frac{\varphi_i}{\prod_{k \neq i,j} F_k(v + c_k)} \tag{28}$$

for all $i \in \mathcal{I} \setminus \{j\}$. For all $\mathbf{t} \notin K^{\emptyset}(v) \cup K^j(v)$, we define $\mathbf{p}'$ by applying rules *1-2* of Definition 4.

We proceed to argue that $\mathbf{p}'$ is feasible, i.e. $p_i'(\mathbf{t}) \in [0,1]$ for all i and all $\mathbf{t}$, and $\sum_i p_i'(\mathbf{t}) \leq 2$ for all $\mathbf{t}$. Note that by definition of $\Phi_j(v, v')$, we have

$$\prod_{k \neq j} F_k(v + c_k) = \sum_{i \in \mathcal{I} \setminus \{j\}} \varphi_i F_i(v + c_i) \geq \varphi_i F_i(v + c_i).$$

Thus, $p_i'(\mathbf{t}) \leq 1$ for all $i \in \mathcal{I} \setminus \{j\}$ and all $\mathbf{t} \in K^{\emptyset}(v) \cup K^j(v)$. Obviously, for agent j on $K^{\emptyset}(v) \cup K^j(v)$ and all agents on $\mathbf{T} \setminus \left[K^{\emptyset}(v) \cup K^j(v)\right]$, $p_j'(\mathbf{t})$ or $p_i'(\mathbf{t})$ is either 0 or 1.

For each $\mathbf{t} \in K^{\emptyset}(v) \cup K^{j}(v)$, we know that $p'_j(\mathbf{t}) = 1$ and that

$$\begin{aligned}\sum_{i \in \mathcal{I}\setminus\{j\}} p'_i(\mathbf{t}) = &\sum_{i \in \mathcal{I}\setminus\{j\}} \frac{\varphi_i}{\prod_{k \neq i,j} F_k(v+c_k)} \\ =&\frac{1}{\prod_{k\neq j} F_k(v+c_k)} \sum_{i \in \mathcal{I}\setminus\{j\}} \varphi_i F_i(v+c_i) \\ =&1,\end{aligned}$$

where the last equality follows from the definition of $\Phi_j(v, v')$. Therefore, $\mathbf{p}'$ is feasible on $K^{\emptyset}(v) \cup K^{j}(v)$. It is straightforward to verify that $\mathbf{p}'$ is feasible on $\mathbf{T} \setminus \left[K^{\emptyset}(v) \cup K^{j}(v)\right]$.

Part 2. Verification We proceed to show that $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$. To simplify notations, we denote $\mathbf{E}_{\mathbf{t}_{-i}}[p_i(t_i, \mathbf{t}_{-i})]$ by $\hat{p}_i(t_i)$. For agent j, if $t_j - c_j \in [\underline{t}_j - c_j, v']$, then by the construction based on Definition 4 we have

$$\hat{p}'_j(t_j) = \prod_{k \in \mathcal{I}\setminus\{j\}} F_k(v'+c_k) + \sum_{i \in \mathcal{I}\setminus\{j\}} \left[1 - F_i(v'+c_i)\right] \prod_{k \in \mathcal{I}\setminus\{i,j\}} F_k(v'+c_k) = \varphi_j^{\mathbf{p}},$$

where the second equality follows from the definition of v'. If $t_j - c_j \in (v', \bar{v}]$, then

$$\begin{aligned}\hat{p}'_j(t_j) = &\prod_{k \in \mathcal{I}\setminus\{j\}} F_k(t_j - c_j + c_k) + \sum_{i \in \mathcal{I}\setminus\{j\}} [1 - F_i(t_j - c_j + c_i)] \prod_{k \in \mathcal{I}\setminus\{i,j\}} F_k(t_j - c_j + c_k) \\ \geq &\prod_{k \in \mathcal{I}\setminus\{j\}} F_k(v'+c_k) + \sum_{i \in \mathcal{I}\setminus\{j\}} \left[1 - F_i(v'+c_i)\right] \prod_{k \in \mathcal{I}\setminus\{i,j\}} F_k(v'+c_k) \\ =&\varphi_j^{\mathbf{p}},\end{aligned}$$

where the inequality follows because the right-hand-side expression is increasing in $t_j - c_j$.

For each $i \in \mathcal{I} \setminus \{j\}$ and each t_i with $\underline{t}_i - c_i \leq t_i - c_i \leq v$, by (28), we know that $\hat{p}'_i(t_i) = \varphi_i^{\mathbf{p}}$, i.e. rule *3* is satisfied. For each $i \in \mathcal{I} \setminus \{j\}$ and each t_i with $v \leq t_i - c_i \leq v'$, agent i obtains one good if and only if i has the highest or the second highest net value in $K^{\emptyset}(v') \cup K^{j}(v')$. Thus,

$$\begin{aligned}\hat{p}'_i(t_i) = &\prod_{k \in \mathcal{I}\setminus\{i\}} F_k(t_i - c_i + c_k) + [1 - F_j(t_i - c_i + c_j)] \prod_{k \in \mathcal{I}\setminus\{i,j\}} F_k(t_i - c_i + c_k) \\ = &\prod_{k \in \mathcal{I}\setminus\{i,j\}} F_k(t_i - c_i + c_k) \\ \geq &\prod_{k \in \mathcal{I}\setminus\{i,j\}} F_k(v + c_k) \\ \geq &\varphi_i^{\mathbf{p}}.\end{aligned}$$

The argument for t_i with $v' < t_i - c_i \leq \bar{v}$ is similar to that for agent j and thus omitted. To sum up, we have $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$, which also implies that $\mathbf{p}'$ is a j-threshold mechanism with thresholds (v, v'). □

C.4 Proofs for Section 4.4

The lemma below will be used to prove Proposition 1. Its proof is standard and, thus, is omitted.

Lemma 9. *The principal's problem (8)-(9) is equivalent to*

$$\max_{\substack{\varphi_i \in [0,1], \forall i \\ \sum_i \varphi_i \leq 2.}} \quad \max_{\substack{p_i : \mathbf{t} \to [0,1], \forall i \\ \sum_i p_i(\mathbf{t}) \leq 2, \quad \forall \mathbf{t} \in \mathbf{T}.}} \quad \mathbf{E}_{\mathbf{t}}\left[\sum_i p_i(\mathbf{t})(t_i - c_i) + \sum_i \varphi_i c_i\right] \tag{29}$$

$$\text{subject to} \quad \mathbf{E}_{\mathbf{t}_{-i}}\left[p_i(t_i, \mathbf{t}_{-i})\right] \geq \varphi_i, \quad \forall t_i, \quad \forall i. \tag{30}$$

Proof of Proposition 1. We first show that if an optimal mechanism $\mathbf{p}$ lies in P_j for some $j \in \mathcal{I} \cup \{\emptyset\}$, then it is essentially a j-threshold mechanism. Then we argue that optimal j-threshold mechanisms must be essentially (randomizations of) extreme j-threshold mechanisms, which completes the proof of the proposition. Let $\mathbf{p}'$ denote the corresponding threshold mechanism of $\mathbf{p}$, in the sense that $\boldsymbol{\varphi}^{\mathbf{p}'} = \boldsymbol{\varphi}^{\mathbf{p}}$. To simplify notations, we denote $\mathbf{E}_{\mathbf{t}_{-i}}[p_i(t_i, \mathbf{t}_{-i})]$ by $\hat{p}_i(t_i)$. In what follows, we prove only the case of $j \in \mathcal{I}$ and omit the case of $j = \emptyset$, since the argument for the latter is similar to that for the former and much simpler.

Part 1. *If $\mathbf{p} \in P_j(v, v')$ is optimal, then $\mathbf{p}$ is essentially a j-threshold mechanism.*

We proceed to show that (i) for almost all $t_j < v' + c_j$, $\hat{p}_j(t_j) = \hat{p}'_j(t_j)$, and, moreover, for each $i \neq j$ and almost all $t_i < v + c_i$, $\hat{p}_i(t_i) \geq \hat{p}'_i(t_i)$; (ii) for almost all $t_j > v' + c_j$, $\hat{p}_j(t_j) \leq \hat{p}'_j(t_j)$, and, moreover, for each $i \neq j$ and almost all $t_i > v + c_i$, $\hat{p}_i(t_i) \leq \hat{p}'_i(t_i)$; (iii) for each $i \in \mathcal{I}$ and almost all $t_i \in T_i$, $\hat{p}_i(t_i) = \hat{p}'_i(t_i)$; and finally (iv) rules 1 and 2 in Definition 4 are both satisfied. Then, $\mathbf{p}$ is essentially a j-threshold mechanism.

Step (i). *For almost all $t_j < v' + c_j$, $\hat{p}_j(t_j) = \hat{p}'_j(t_j)$, and, moreover, for each $i \neq j$ and almost all $t_i < v + c_i$, $\hat{p}_i(t_i) \geq \hat{p}'_i(t_i)$.*

Consider $i \neq j$. For almost all $t_i < v + c_i$, we argue $\hat{p}_i(t_i) \geq \hat{p}'_i(t_i)$. Firstly, since $\mathbf{p}$ is optimal, it is feasible in the sense that it satisfies (30), i.e. for each i and almost all t_i, we have $\hat{p}_i(t_i) \geq \varphi_i^{\mathbf{p}}$. Secondly, since $\mathbf{p}'$ is the canonical mechanism of $\mathbf{p}$, we know that $\varphi_i^{\mathbf{p}'} = \varphi_i^{\mathbf{p}}$. Finally, since $\mathbf{p}'$ is a j-threshold mechanism, we know that $\hat{p}'(t_i) = \varphi_i^{\mathbf{p}'}$ for all $t_i < v + c_i$. Therefore, $\hat{p}_i(t_i) \geq \hat{p}'_i(t_i)$ for almost all $t_i < v + c_i$. For similar reasons, we know that for almost all $t_j < v' + c_j$, $\hat{p}_j(t_j) \geq \hat{p}'_j(t_j)$.

Suppose to the contrary that $\hat{p}_j(t_j) > \hat{p}'_j(t_j)$ for a positive measure $D_j \subseteq \{t_j \in T_j : t_j < v' + c_j\}$.

Claim 1. *For a positive measure subset of $t_j < v' + c_j$ we have*

$$\int_{(t_j, \mathbf{t}_{-j}) \in K(v')} p_j(t_j, \mathbf{t}_{-j}) dF_{-j}(\mathbf{t}_{-j}) > \int_{(t_j, \mathbf{t}_{-j}) \in K(v')} p'_j(t_j, \mathbf{t}_{-j}) dF_{-j}(\mathbf{t}_{-j}). \tag{31}$$

Proof. Rewrite the inequality $\hat{p}_j(t_j) > \hat{p}'_j(t_j)$ as

$$\begin{aligned}
&\int_{(t_j, \mathbf{t}_{-j}) \in K(v')} p_j(t_j, \mathbf{t}_{-j}) dF_{-j}(\mathbf{t}_{-j}) + \int_{(t_j, \mathbf{t}_{-j}) \in \mathbf{T} \setminus K(v')} p_j(t_j, \mathbf{t}_{-j}) dF_{-j}(\mathbf{t}_{-j}) \\
> &\int_{(t_j, \mathbf{t}_{-j}) \in K(v')} p'_j(t_j, \mathbf{t}_{-j}) dF_{-j}(\mathbf{t}_{-j}) + \int_{(t_j, \mathbf{t}_{-j}) \in \mathbf{T} \setminus K(v')} p'_j(t_j, \mathbf{t}_{-j}) dF_{-j}(\mathbf{t}_{-j}).
\end{aligned} \tag{32}$$

We claim that for almost all $t_j \in D_j$,

$$\int_{(t_j,\mathbf{t}_{-j})\in\mathbf{T}\setminus K(v')} p_j(t_j,\mathbf{t}_{-j})dF_{-j}(\mathbf{t}_{-j}) \leq \int_{(t_j,\mathbf{t}_{-j})\in\mathbf{T}\setminus K(v')} p'_j(t_j,\mathbf{t}_{-j})dF_{-j}(\mathbf{t}_{-j}). \tag{33}$$

Suppose not. Then, there exists a positive measure subset of $(t_j,\mathbf{t}_{-j}) \in \mathbf{T}\setminus K(v')$ such that $p_j(t_j,\mathbf{t}_{-j}) > p'_j(t_j,\mathbf{t}_{-j})$. Note that at every type profile in $\mathbf{T} \setminus K(v')$ such that $t_j < v' + c_j$, $t_j - c_j$ is never among the highest two net values. Thus, we can improve $\mathbf{p}$ by moving some assignment probability from agent j to the agents with the two highest net values. Since $\hat{p}_j(t_j) > \hat{p}'_j(t_j) \geq \varphi_j^{\mathbf{p}}$, moving a small enough amount of assignment probability this way is feasible. Since this would give a mechanism strictly better for the principal than $\mathbf{p}$ and $\mathbf{p}$ is optimal, we know this is impossible. Hence, up to sets of measure zero of t_j, (33) holds. Therefore, (32) implies that for a positive measure subset of $t_j < v' + c_j$, (31) holds. This completes the proof of the claim.

Note that within $K(v')$, $p'_j(t_j,\mathbf{t}_{-j}) = 1$ by definition. Therefore, (31) implies that

$$\int_{(t_j,\mathbf{t}_{-j})\in K(v')} p_j(t_j,\mathbf{t}_{-j})dF_{-j}(\mathbf{t}_{-j}) > \int_{(t_j,\mathbf{t}_{-j})\in K(v')} dF_{-j}(\mathbf{t}_{-j})$$

for a positive measure subset of $t_j < v' + c_j$. This in turn implies that $p_j(t_j,\mathbf{t}_{-j}) > 1$ for a positive measure subset of type profiles, which contradicts the feasibility of $\mathbf{p}$. Therefore, for almost all $t_j < v' + c_j$, we have $\hat{p}_j(t_j) = \hat{p}'_j(t_j)$.

Step (ii). *For almost all $t_j > v' + c_j$, $\hat{p}_j(t_j) \leq \hat{p}'_j(t_j)$, and, moreover, for each $i \neq j$ and almost all $t_i > v + c_i$, $\hat{p}_i(t_i) \leq \hat{p}'_i(t_i)$.*

Suppose to the contrary that for some $i \neq j$, we have $\hat{p}_i(t_i) > \hat{p}'_i(t_i)$ for a positive measure subset of t_i such that $t_i - c_i > v$. Since a sufficient condition for agent i to receive one good under $\mathbf{p}'$ when $t_i - c_i > v$ is that she has one of the two highest net values, we know that there is a positive measure subset of $\mathbf{t}_{-i}$ such that agent i receives a good under $\mathbf{p}$ even though her net value is not one of the two highest net values. Since $\hat{p}_i(t_i) > \hat{p}'_i(t_i) \geq \varphi_i^{\mathbf{p}}$, on such profiles $(t_i, \mathbf{t}_{-i})$ we can (a) improve $\mathbf{p}$ by moving some assignment probability from agent i to the two agents with the highest net values and (b) maintain the constraint (30). Since this would give a mechanism strictly better for the principal than $\mathbf{p}$ and $\mathbf{p}$ is optimal, we know this is impossible. Hence, up to sets of measure zero, $\hat{p}_i(t_i) \leq \hat{p}'_i(t_i)$ all t_i such that $t_i - c_i > v$. For similar reasons, we know that for almost all $t_j > v' + c_j$, $\hat{p}_j(t_j) \leq \hat{p}'_j(t_j)$.

Step (iii). *For each $i \in \mathcal{I}$ and almost all $t_i \in T_i$ $\hat{p}_i(t_i) = \hat{p}'_i(t_i)$.*

By **Step (i)** and **Step (ii)**, we know that for agent j, for almost all $t_j < v' + c_j$, $\hat{p}_j(t_j) = \hat{p}'_j(t_j)$, and for almost all $t_j > v' + c_j$, $\hat{p}_j(t_j) \leq \hat{p}'_j(t_j)$. Moreover, for each $i \neq j$, (a) for almost all $t_i < v + c_i$, $\hat{p}_i(t_i) \geq \hat{p}'_i(t_i)$; (b) for almost all $t_i > v + c_i$, $\hat{p}_i(t_i) \leq \hat{p}'_i(t_i)$. An almost identical argument as in Lipman (2015) can show that $\hat{p}_i(t_i) = \hat{p}'_i(t_i)$.

Step (iv). *Rules 1 and 2 in Definition 4 are both satisfied.*

Clearly, up to sets of measure zero, we have for all $i \neq j$ that $\hat{p}_i(t_i) = \varphi_i^{\mathbf{p}}$ for all $t_i < v + c_i$. Suppose for some agent $i \neq j$ and a positive measure set of $t_i > v + c_i$ that $\hat{p}_i(t_i) = \hat{p}'_i(t_i)$ but that the allocation for some profiles $\mathbf{t}_{-i}$ (positive measure) differ across p_i and p'_i given t_i. Since by Definition 4, agent i with $t_i > v + c_i$ receives one good under $\mathbf{p}'$ if and only if he has one of the two highest net values, we know that on such profiles $(t_i, \mathbf{t}_{-i})$, the mechanism $\mathbf{p}$ can be improved. Particularly, note that for all $t_i > v + c_i$, we have $\hat{p}'_i(t_i) > \varphi_i^{\mathbf{p}}$, which follows from our construction and the atomlessness of distributions, which implies that $\hat{p}_i(t_i) > \varphi_i^{\mathbf{p}}$ for almost all $t_i > v + c_i$. As a result, we can (i) improve $\mathbf{p}$ by moving some assignment probability from agent i to the two agents who have the highest net values and (ii) maintain the constraint (30). Since this would give a mechanism strictly better for the principal than $\mathbf{p}$ and $\mathbf{p}$ is optimal, we know this is impossible. Hence, for all $i \neq j$ and almost all $t_i > v + c_i$, $p_i(\mathbf{t}) = p'_i(\mathbf{t})$.

Note that at every $\mathbf{t} \in \mathbf{T} \setminus K(v')$ such that $t_j < v' + c_j$, $t_j - c_j$ is never among the highest two net values. Thus, to satisfy $\hat{p}_j(t_j) \geq \varphi_j^{\mathbf{p}}$, we must have $p_j(\mathbf{t}) = 1$ for all type profiles in $K(v')$ such that $t_j < v' + c_j$, i.e. $p_j(\mathbf{t}) = p'_j(\mathbf{t})$. For similar reasons as in the argument for agents $i \neq j$, we know that for almost all type profiles such that $t_j > v' + c_j$, $p_j(\mathbf{t}) = p'_j(\mathbf{t})$. Therefore, Rules 1 and 2 are both satisfied, and thus $\mathbf{p}$ is essentially a j-threshold mechanism.

Part 2. *An optimal mechanism in P_j must be essentially randomizations of extreme j-mechanisms.*

The principal's payoff from a threshold mechanism is determined by a vector $\boldsymbol{\varphi}$. Particularly, it is

$$\begin{aligned}
&\varphi_j c_j + \int_{\underline{t}_j}^{v'+c_j} \varphi_j (t_j - c_j) dF_j(t_j) \\
&+ \int_{v'+c_j}^{\bar{t}_j} \begin{bmatrix} \prod_{k \neq j} F_k(t_j - c_j + c_k) + \\ \sum_{m \neq j} [1 - F_m(t_j - c_j + c_m)] \prod_{k \neq j,m} F_k(t_j - c_j + c_m) \end{bmatrix} (t_j - c_j) dF_j(t_j) \\
&+ \sum_{i \in \mathcal{I} \setminus \{j\}} \varphi_i c_i + \sum_{i \in \mathcal{I} \setminus \{j\}} \int_{\underline{t}_i}^{v+c_i} \varphi_i (t_i - c_i) dF_i(t_i) \\
&+ \sum_{i \in \mathcal{I} \setminus \{j\}} \int_{v+c_i}^{v'+c_i} \begin{bmatrix} \prod_{k \neq i} F_k(t_i - c_i + c_k) + \\ [1 - F_j(t_i - c_i + c_j)] \prod_{k \neq i,j} F_k(t_i - c_i + c_m) \end{bmatrix} (t_i - c_i) dF_i(t_i) \\
&+ \sum_{i \in \mathcal{I} \setminus \{j\}} \int_{v'+c_i}^{\bar{t}_i} \begin{bmatrix} \prod_{k \neq i} F_k(t_i - c_i + c_k) + \\ \sum_{m \neq i} [1 - F_m(t_i - c_i + c_m)] \prod_{k \neq i,m} F_k(t_i - c_i + c_m) \end{bmatrix} (t_i - c_i) dF_i(t_i).
\end{aligned}$$

It is obvious that the payoff is linear in $\boldsymbol{\varphi}$. Thus, we only need to focus on the extreme points of $\Phi_j(v, v')$, which are characterized in Lemma 4. □

C.5 Proofs for Section 4.5

The proof of Lemma 7 is similar to that of Theorem 2 in Ben-Porath et al. (2014). Particularly, we first calculate the difference in payoffs brought by two alternatives: v_j^* and another v' such that $v' > v_j^*$. Let

x be the second highest net value. Without loss of generality, this means that there exists k such that $t_k - c_k = x$ and that $|i : t_i - c_i > x| = 1$. The superiority of v_j^* is obtained by examining the principal's payoffs under v_j^* and v' in three subdomains such that $x < v_j^* < v'$, $v_j^* < x < v'$, or $v_j^* < v' < x$. Symmetrically, v_j^* is superior than any $v' < v_j^*$. Thus, v_j^* is the unique optimal threshold. We refer the readers to Ben-Porath et al. (2014) for details.

Here we present a proof of Lemma 8 that is interpretable. Readers who are interested could directly decompose the principal's objective function and arrive at the same conclusion.

Proof of Lemma 8. We start with a benchmark mechanism and analyze the difference between each of the candidate mechanisms and the benchmark mechanism. Let the benchmark mechanism be as follows: for each type profile, the two goods always go to the two agents with the two highest net values. An agent is (partially) checked if and only if he (partially) gets a good.

To derive either of the two candidate mechanisms, we conduct the following modification to the benchmark mechanism, which is the same for two candidates.[25] For every type profile $\mathbf{t} \in K^{\emptyset}(v_1^*) \cup K^j(v_1^*)$, agent 1 gets one good without been checked; for every type profile $\mathbf{t} \in K^{\emptyset}(v_j^*) \cup K^1(v_j^*)$, agent j gets one good without been checked. Particularly, this is done by three steps:

1. Within the region $K^{\emptyset}(v_1^*) \cup K^j(v_1^*)$, we let agent 1 replace the agent who has the second highest net value when agent 1 did not get one good in the benchmark mechanism.
2. Within the region $[K^{\emptyset}(v_j^*) \cup K^1(v_j^*)] \cap [K^{\emptyset}(v_1^*) \cup K^j(v_1^*)]$, we let agent j replaces the agent who has the highest net value when agent j did not get one good in the benchmark mechanism.
3. Within the region $[K^{\emptyset}(v_j^*) \cup K^1(v_j^*)] \setminus [K^{\emptyset}(v_1^*) \cup K^j(v_1^*)]$ we let agent j replace the agent who has the second highest net value when agent j did not get one good in the benchmark mechanism.

Now we deal with the difference between $(1, v_1^*; j, v_j^*)$ and $(j, v_j^*; 1, v_1^*)$. First, to derive $(1, v_1^*; j, v_j^*)$, we consider the following regions: for each $m \neq 1, j$, let $\bar{K}^m(v, 1)$ be the "extended finger m in the direction of agent 1" such that

$$\bar{K}^m(v, 1) := \left\{ \mathbf{t} \in \mathbf{T} : t_m > v \text{ and } \max_{k \neq 1, m} t_k - c_k < v \right\}.$$

Then $(1, v_1^*; j, v_j^*)$ is derived by letting agent 1 get one good in $\bar{K}^m(v_1^*, 1)$, for all $m \neq 1, j$, without being checked. This is done by replacing the agent who has the second highest net value whenever agent 1 did not get one good in the benchmark mechanism. In the extended finger $m \neq 1, j$, the payoff

[25] The overlapping description between the candidate mechanisms and the benchmark mechanism makes the presentation simpler without changing the conclusion. The same remark applies to the rest of the proof. Readers may find the explanations attached to relaxed subproblems (13) and (14) useful to better understand the modifications made to the benchmark mechanism in this proof.

change from the benchmark mechanism to $(1, v_1^*; j, v_j^*)$ is given by

$$\begin{aligned}
&[1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq 1,m} F_k(v_1^* + c_k)\right] c_1 \\
&+ [1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq 1,m} F_k(v_1^* + c_m)\right] \int_{\underline{t}_1}^{v_1^*+c_1} (t_1 - c_1) dF_1(t_1) \\
&- [1 - F_m(v_1^* + c_m)] \sum_{i \neq m} \int_{\underline{t}_i}^{v_1^*+c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k)\right] (t_i - c_i) dF_i(t_i),
\end{aligned} \tag{34}$$

where the first term is the saved checking cost and the difference between the second term and the third term is the gain from changing assignment.

Second, to derive $(j, v_j^*; 1, v_1^*)$, we consider the following regions: for each $m \neq 1, j$, let $\bar{K}^m(v, j)$ be the "extended finger m in the direction of agent j" such that

$$\bar{K}^m(v, j) := \left\{ \mathbf{t} \in \mathbf{T} : t_m > v \text{ and } \max_{k \neq j,m} t_k - c_k < v \right\}.$$

Then $(j, v_j^*; 1, v_1^*)$ is derived by letting agent j get one good in $\bar{K}^m(v_j^*, j)$, for all $m \neq 1, j$, without being checked. This is done by (i) replacing the agent who has the second highest net value whenever agent j did not get one good in the benchmark mechanism, and (ii) replacing the agent who has the highest net value whenever $t_m - c_m \in [v_j^*, v_1^*]$ and agent j did not get one good in the benchmark mechanism. In the extended finger $m \neq 1, j$, the payoff change from the benchmark mechanism to $(j, v_j^*; 1, v_1^*)$ is given by

$$\begin{aligned}
&[1 - F_m(v_j^* + c_m)] \left[\prod_{k \neq j,m} F_k(v_j^* + c_k)\right] c_j \\
&+ [1 - F_m(v_j^* + c_m)] \left[\prod_{k \neq j,m} F_k(v_j^* + c_m)\right] \int_{\underline{t}_j}^{v_j^*+c_j} (t_j - c_j) dF_j(t_j) \\
&- \int_{v_j^*+c_m}^{v_1^*+c_m} \left[\prod_{k \neq m} F_k(t_m - c_m + c_k)\right] (t_m - c_m) dF_m(t_m) \\
&- [1 - F_m(v_1^* + c_m)] \sum_{i \neq m} \int_{\underline{t}_i}^{v_j^*+c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k)\right] (t_i - c_i) dF_i(t_i),
\end{aligned} \tag{35}$$

where the first term is the saved checking cost, the second term is the payoff from the new assignment rule, the third term is the assignment loss for type profiles with $t_m - c_m \in [v_j^*, v_1^*]$ and the last terms is the assignment loss for type profiles with $t_m - c_m > v_1^*$.

To show that $(1, v_1^*; j, v_j^*)$ delivers a higher payoff than $(j, v_j^*; 1, v_1^*)$, it suffices to show that (34)

> (35) for every $m \neq 1, j$. Note that the first two terms of (34) can be simplified as

$$\begin{aligned}
&[1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq 1,m} F_k(v_1^* + c_k) \right] c_1 \\
&+ [1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq 1,m} F_k(v_1^* + c_m) \right] \int_{\underline{t}_1}^{v_1^* + c_1} (t_1 - c_1) dF_1(t_1) \\
= &[1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq m} F_k(v_1^* + c_k) \right] \left[\frac{\int_{\underline{t}_1}^{v_1^* + c_1} (t_1 - c_1) dF_1(t_1)}{F_1(v_1^* + c_1)} + \frac{c_1}{F_1(v_1^* + c_1)} - c_1 \right] \\
= &[1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq m} F_k(v_1^* + c_k) \right] v_1^*,
\end{aligned}$$

where the second equality follows from the definition of v_1^*, i.e. (6). Then (34) becomes

$$\begin{aligned}
&[1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq m} F_k(v_1^* + c_k) \right] v_1^* \\
&- [1 - F_m(v_1^* + c_m)] \sum_{i \neq m} \int_{\underline{t}_i}^{v_1^* + c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k) \right] (t_i - c_i) dF_i(t_i) \\
= &[1 - F_m(v_1^* + c_m)] \left\{ \left[\prod_{k \neq m} F_k(v_1^* + c_k) \right] v_1^* - \sum_{i \neq m} \int_{\underline{t}_i}^{v_1^* + c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k) \right] (t_i - c_i) dF_i(t_i) \right\},
\end{aligned}$$

where the difference is exactly the gain from replacing all net values $t_i - c_i$, $i \neq m$, in the region

$$\left\{ \mathbf{t} \in \mathbf{T} : \quad t_m - c_m > v_1^* \quad \text{and} \quad \max_{i \neq m} t_i - c_i < v_1^* \right\} \tag{36}$$

with their upper bound v_1^*.

Similarly, (35) becomes

$$\begin{aligned}
&\left[1 - F_m(v_j^* + c_m)\right] \left[\prod_{k \neq m} F_k(v_j^* + c_k) \right] v_j^* - \int_{v_j^* + c_m}^{v_1^* + c_m} \left[\prod_{k \neq m} F_k(t_m - c_m + c_k) \right] (t_m - c_m) dF_m(t_m) \\
&- [1 - F_m(v_1^* + c_m)] \sum_{i \neq m} \int_{\underline{t}_i}^{v_j^* + c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k) \right] (t_i - c_i) dF_i(t_i).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{v_j^* + c_m}^{v_1^* + c_m} \left[\prod_{k \neq m} F_k(t_m - c_m + c_k) \right] (t_m - c_m) dF_m(t_m) \\
\geq &\left[F_m(v_1^* + c_m) - F_m(v_j^* + c_m)\right] \left[\prod_{k \neq m} F_k(v_j^* + c_k) \right] v_j^*,
\end{aligned}$$

we know that

$$
\begin{aligned}
(35) \leq & \left[1 - F_m(v_j^* + c_m)\right] \left[\prod_{k \neq m} F_k(v_j^* + c_k)\right] v_j^* - \left[F_m(v_1^* + c_m) - F_m(v_j^* + c_m)\right] \left[\prod_{k \neq m} F_k(v_j^* + c_k)\right] v_j^* \\
& - [1 - F_m(v_1^* + c_m)] \sum_{i \neq m} \int_{\underline{t}_i}^{v_j^* + c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k)\right] (t_i - c_i) dF_i(t_i) \\
= & [1 - F_m(v_1^* + c_m)] \left[\prod_{k \neq m} F_k(v_j^* + c_k)\right] v_j^* \\
& - [1 - F_m(v_1^* + c_m)] \sum_{i \neq m} \int_{\underline{t}_i}^{v_j^* + c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k)\right] (t_i - c_i) dF_i(t_i) \\
= & [1 - F_m(v_1^* + c_m)] \left\{ \left[\prod_{k \neq m} F_k(v_j^* + c_k)\right] v_j^* - \sum_{i \neq m} \int_{\underline{t}_i}^{v_j^* + c_i} \left[\prod_{k \neq i,m} F_k(t_i - c_i + c_k)\right] (t_i - c_i) dF_i(t_i) \right\},
\end{aligned}
$$

where the difference is exactly the gain from replacing all net values $t_i - c_i$, $i \neq m$, in the region

$$
\left\{ \mathbf{t} \in \mathbf{T} : \quad t_m - c_m > v_1^* \quad \text{and} \quad \max_{i \neq m} t_i - c_i < v_j^* \right\} \tag{37}
$$

to their upper bound v_j^*. Since (36) is a superset of (37), we know that (34) > (35), as desired. □