Learning by Matching*

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Abstract

This paper studies stability notions and matching processes in the job market with incomplete information on the workers' side. Each agent is associated with a type, which determines their payoffs from a match. Moreover, firms' information structure is described by partitions over possible worker type profiles. With this firm-specific information, we propose stability notions which, in addition to requiring individual rationality and no blocking pair, captures the idea that the absence of rematching conveys no further information. When an allocation is not stable under the status quo information structure, a new pair of an allocation and an information structure will be derived. We show that starting from an arbitrary allocation and an arbitrary information structure, the process of allowing randomly chosen blocking pairs to rematch, accompanied by information updating, will converge with probability one to an allocation that is stable under the updated information structure. Our results are robust with respect to various alternative learning patterns.

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1 Introduction

Matching is one of the important functions of markets (Roth (2008)). This paper focuses on stability and matching processes in the job-market setting.[1] We depart from the prevailing assumption of two-sided matching theory that information is complete, i.e., that the characteristics of all market participants are common knowledge. In particular, we study incomplete information on the worker side. We first describe how firms form and update their possibilistic information about workers' types, and propose a incomplete-information stability notion that allows for flexible information structure. Then we show that a random matching process converges to an allocation that is stable under the updated information structure, with probability one.

Stability under complete information requires individual rationality, such that each agent has a nonnegative payoff, and no blocking pair; in the present context, no blocking pair means that no worker-firm pair would prefer being matched with each other at a certain wage to staying with the current matching.[2] In contrast, in a job market with incomplete information on the workers' side, a typical firm may not know the types of their potential employees which reflect their productivity. Without this information, however, the firm does not know if it would prefer another worker to its current employee. As a result, the notions of blocking and stability in the complete-information environment is no longer appropriate.

Following Liu et al. (2014) (LMPS for short), we assume that firms that are uncertain about their potential employees' types care about the worst possibility, and that a firm can observe the type of its own employee. Unlike LMPS, however, we describe the firms' heterogeneous information by a profile of partition over the set of type profiles of the workers. Given an information structure, we propose a stability notion which extends the notion of stable matching with incomplete information proposed by LMPS.[3] In our setting, a state of the market consists of an allocation (i.e., a matching together with a prevailing wage profile) and an information structure. A state is stable if the allocation (i) is individually rational and (ii) admits no blocking pair with respect to the information structure, and (iii) the absence of rematching conveys no further information to the firms. The last requirement, in particular, formalizes the "informational stability" which is specific to the incomplete-information setting.

[1] Stable matchings have been connected to both equity and efficiency in resource allocation, two of the most important objectives in economics. See Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) for how stability implies the elimination of justified envy, a basic fairness axiom. See Shapley and Shubik (1971) and Liu et al. (2014) for how stability leads to efficiency.

[2] We use the job-market setting (with transferable utility) to facilitate the comparison between our stability notion and that of Liu et al. (2014). Nevertheless, our convergence result (Theorems 2-2′), can be established without difficulty in models with non-transferable utility, such as the ex ante stability notion of Bikhchandani (2017).

[3] Specifically, our notion of stability coincides with the notion proposed by LMPS when the only source of firms' heterogeneous information is the fact that each firm can observe the type of its own employee. (See subsection 3.4 for details).

Equipped with the notion of stability, we then study a matching process which mimics the behavior of individuals searching for a desired job, a school or a life partner. Indeed, if a worker and a firm find themselves better off being matched with each other than in maintaining the status quo, they will form a match together to make an improvement. The new matching may again admit a blocking pair and thus a rematching opportunity, which results in another new matching, and so on. One prominent question is whether such a process finally stops at a stable matching.[4]

When information is incomplete, each observation of rematching or no rematching along the matching process provides information to the firms. Consequently, even if no rematching is observed under an information structure, updated information may provide some rematching opportunity. Therefore, a matching process is necessarily associated with an information updating process in which firms draw inferences from each observation.

In this information updating process, information heterogeneity arises endogenously. Specifically, we classify the firms' observations into the following three groups: where a rematching is not observed, where a rematching of some pair is observed, and where the firm directly observes the type of the worker within each pair. When a firm updates its information along with each observation, that refines each firm's information partition, potentially in different ways for different firms because of rematching and direct observation. In this sense, modeling firms' heterogeneous information as we do becomes necessary when we study matching processes.

For an arbitrary initial market state, this learning and rematching process consists of a sequence of states. We call it a *learning-blocking path*. Our main result shows that when we suitably choose the blocking pairs, a learning-blocking path, after a finite number of rematchings, reaches an allocation which is stable with respect to the updated information. The finite number depends on the number of agents but not the number of worker-type profiles. This construction implies that when blocking pairs are randomly selected to rematch, the resulting learning-blocking path converges to a stable state with probability one. Our result also holds under various alternative learning patterns.[5]

The rest of this section reviews the related literature. Section 2 introduces the model. Section 3 defines stability with incomplete information. Section 4 describes a specific learning pattern along a learning-blocking path and presents our convergence results. The robustness of convergence, the connection between our stability notion and that of LMPS, and the efficiency of stable states are discussed in Section 5. Section 6 concludes.

[4] Knuth (1976) provides an example of a blocking path that admits a cycle, i.e., any matching on the path is not stable. This motivates the study of the convergence of blocking paths. The literature focusing on this question demonstrates that the answer is primarily a positive one. See Roth and Vande Vate (1990), Kojima and Ünver (2008), Klaus and Klijn (2007), Chen et al. (2016) and Fujishige and Yang (2016).

[5] For example, agents may ignore/forget the information conveyed in some observations or draw more sophisticated inference from the observations. See subsection 5.2 for more discussion.

The Related Literature

The seminal model of Gale and Shapley (1962), which studies the marriage market and the college admission market, has been used in many studies of two-sided matching. Many classical theories are surveyed in Roth and Sotomayor (1990) and more recently by, for example, Roth (2008). In this literature, one prevailing assumption is that the information is complete.

Recently, LMPS introduced a notion of incomplete-information stability. Our notion of stability coincides with the notion proposed by LMPS when the only source of firms' heterogeneous information is the fact that each firm can observe the type of its own employee. More precisely, take an incomplete-information stable matching outcome in their setting, where an outcome consists of an allocation and a worker-type profile. The stability of LMPS can be interpreted as a result of an iteration that starts with a state consisting of the stable matching outcome and a *particular* partition profile. For each firm, the particular partition profile distinguishes only the type of its own employee. That is, each firm puts all worker-type profiles with the same type for its employee into the same cell; the iteration refines the initial partition profile using the information conveyed by the fact that the state is not blocked. The iteration starting with the *particular* partition profile finally stops at a stable state in our setting. In contrast to LMPS, we allow for *flexible* partition profiles in stable states. Our flexible information structure for firms facilitates our study of the matching process.

The stability notion of Bikhchandani (2017) is similar to that of LMPS but focuses on non-transferable utility and Bayesian stability, instead of the worst case desideratum. Unlike LMPS and Bikhchandani (2017), Pomatto (2015) uses a non-coorperative-game approach. He considers a matching game and derives the same incomplete-information stable outcomes as LMPS by using forward-induction reasoning. Instead of focusing on a pre-existing matching outcome, Chakraborty et al. (2010) study matching mechanisms that produce stable matchings. They consider college admission markets in which the students' characteristics are unobservable for the colleges but the colleges may receive signals about these characteristics. They propose and identify a stable matching *mechanism* which specifies a matching and how much information to reveal to the agents, given each reported signal profile with an information structure. They also make clear that stability of a matching mechanism should depend on the agents' information structure.[6] In a similar vein, we define a stable allocation with respect to different information structures of the agents.[7]

[6]To the best of our knowledge, Chakraborty et al. (2010) is the first paper that incorporates available information into stability notions. Another stream of literature studies incomplete information about other's preferences. See, for example, Roth (1989) and Ehlers and Massó (2007).

[7]Our stability notion is also related to the literature on the core, particularly the core in incomplete-information problems. In our context, a coalition is simply a worker-firm pair. See Wilson (1978), Dutta and Vohra (2005), and the comprehensive discussions in LMPS.

Whether a matching process converges to a stable allocation is known as the problem of finding paths to stability. The first result on paths to stability was shown in marriage markets with complete information by Roth and Vande Vate (1990) (hereafter, RV). Chen et al. (2010) investigate the path to stability in models with complete information and transferable utilities.[8] All these results involve no private types or information updating, whereas both private types and information updating are crucial in the current paper. Our result implies an alternative proof for RV's theorem, as well as for Theorem 1 of Chen et al. (2010). Bikhchandani (2017) discusses the path to stability under a Bayesian notion of stability. In his paper, the final matching outcome of a blocking path is Bayesian-stable *conditional on history*. That is, along a blocking path, agents cannot block with any of their erstwhile partners. However, we do not impose such a restriction here. Lazarova and Dimitrov (2017) also studies paths to stability with incomplete information, under a permissive blocking notion that enables agents to learn the types of other agents on the opposite side of the market as long as there is a possibility of learning by satisfying permissive blocking pairs. However, their approach is not applicable when more conservative blocking notions are adopted, such as the notions in LMPS, Pomatto (2015) and Bikhchandani (2017).

2 The Model

We consider the following setup of matching with incomplete information due to LMPS. The setup generalizes the complete-information matching models studied by Shapley and Shubik (1971) and Crawford and Knoer (1981).

There is a finite set I of workers to be matched with a finite set J of firms. Denote a generic worker by i and a generic firm by j. While each agent's index i or j is publicly observed, the productivity is determined by the agent's *type*. Let $W \subset \mathbb{R}$ be the finite set of worker types and $F \subset \mathbb{R}$ be the finite set of firm types. A type assignment for firms is a mapping $\mathbf{f} : J \to F$, and likewise a type assignment for workers is another mapping $\mathbf{w} : I \to W$. We denote by Ω a set of type assignments for workers, i.e., $\Omega \subset W^I$.

A match between a worker of type $w \in W$ and a firm of type $f \in F$ gives rise to the *worker premuneration value* $\nu_{wf} \in \mathbb{R}$ and *firm premuneration value* $\phi_{wf} \in \mathbb{R}$.[9] The sum of the two premuneration values $\nu_{wf} + \phi_{wf}$ is called the *surplus of the match*. Denote these values by $\nu_{\mathbf{w}(\emptyset),\mathbf{f}(j)}$ for the unmatched worker and $\phi_{\mathbf{w}(i),\mathbf{f}(\emptyset)}$ for the unmatched firm, both of which are set to be zero. The functions $\nu : W \times F \to \mathbb{R}$ and $\phi : W \times F \to \mathbb{R}$ are common knowledge among the agents.

Given a match between worker i (of type $\mathbf{w}(i)$) and firm j (of type $\mathbf{f}(j)$), the worker's

[8] The main result of Chen et al. (2010), partially incorporated into Chen et al. (2016), is the convergence of blocking paths to *competitive equilibrium*, which is stronger than stability. See also Fujishige and Yang (2016).

[9] See Mailath et al. (2013, 2017) for discussions on premuneration values.

payoff and the firm's payoff are, respectively, $\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p$ and $\phi_{\mathbf{w}(i),\mathbf{f}(j)} - p$, where $p \in \mathbb{R}$ is the payment made to worker i by firm j.[10]

A *matching* is a function $\mu : I \to J \cup \{\emptyset\}$, one-to-one on $\mu^{-1}(J)$, that assigns worker i to firm $\mu(i)$, where $\mu(i) = \emptyset$ means that worker i is unemployed and $\mu^{-1}(j) = \emptyset$ means that firm j does not hire any worker. A *payment scheme* $\mathbf{p}$ associated with a matching μ is a vector that specifies a payment $\mathbf{p}_{i,\mu(i)} \in \mathbb{R}$ for each $i \in I$ and $\mathbf{p}_{\mu^{-1}(j),j} \in \mathbb{R}$ for each $j \in J$. To avoid nuisance cases, we associate zero payments with unmatched agents, by setting $\mathbf{p}_{\emptyset j} = \mathbf{p}_{i\emptyset} = 0$. Finally, an *allocation* $(\mu, \mathbf{p})$ consists of a matching μ and an associated payment scheme $\mathbf{p}$. We assume that the entire allocation is publicly observable. Denote by $\mathscr{A}$ the set of allocations.

As in LMPS, we assume that each firm's type (i.e., $\mathbf{f}$) is common knowledge.[11] There is however incomplete informaiton about the worker's type. In particular, it is only common knowledge that the worker type assignment belongs to Ω, each worker knows his own type, and each firm knows her current employee's type. Beyond the public information, each firm may also have her own private information about the worker type assignment. Specifically, for every j, we describe firm j's private information by a partition Π_j over Ω. For any $\mathbf{w} \in \Omega$, write $\Pi_j(\mathbf{w})$ as the element of Π_j that contains $\mathbf{w}$. Each $\mathbf{w}' \in \Pi_j(\mathbf{w}) \in \Pi_j$ is a possible type assignment from firm j's point of view, when the true type assignment is $\mathbf{w}$. Denote the profile of partitions by Π, i.e., $\Pi := (\Pi_1, \ldots, \Pi_{|J|})$, which is assumed to be common knowledge. We illustrate the formulation with the following example.

Example 1. *Consider a job market in which we have two workers and two firms. Particularly, $I = \{\alpha, \beta\}$ and $J = \{a, b\}$. A type assignment for workers in this market is a two dimensional vector, where the first component is the type for α and the second for β. There are three possible type-assignments, i.e., $\Omega = \{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\}$. The details of Ω are given below.*

$$\begin{aligned} \mathbf{w}^1 &= (w_\alpha, w_\beta) \\ \mathbf{w}^2 &= (w'_\alpha, w_\beta) \\ \mathbf{w}^3 &= (w'_\alpha, w'_\beta) \end{aligned} \tag{1}$$

Suppose that $\mathbf{w}^1$ is the true type-assignment. Suppose firm a hires worker α at a salary of 0 and b hires β at 0. In other words, a matching μ is given by $\mu(\alpha) = a$ and $\mu(\beta) = b$, and $\mathbf{p}_{\alpha,a} = \mathbf{p}_{\beta,b} = 0$. Suppose that each firm observes the type of its employee and knows nothing else. We proceed to describe firms' information about the true type-assignment. Firm a observes the type of its employee, which is w_α. With this information, firm a

[10]If we adopt the intuition that salaries must be rounded to the nearest dollar, penny, or mill, the analysis in this section will go through without any extra difficulty. This more practical restriction will be imposed in Section 4, where we study a matching process.

[11]It is certainly important to study matching markets with two-sided incomplete information, which involves subtle formulation of the agents' higher-order reasoning. See Chen and Hu (2017) for details.

knows that the true type-assignment is $\mathbf{w}^1$*, instead of any one of* $\mathbf{w}^2$ *and* $\mathbf{w}^3$*. Similarly, firm b observes the type of* β*, which is* w_β*. With this information, firm b knows that the true type-assignment must be one of* $\mathbf{w}^1$ *and* $\mathbf{w}^2$*, instead of* $\mathbf{w}^3$*. Using our notation, firms' information is summarized by the following partition profile.*

$$\Pi_a = \big\{\{\mathbf{w}^1\}, \{\mathbf{w}^2, \mathbf{w}^3\}\big\}$$
$$\Pi_b = \big\{\{\mathbf{w}^1, \mathbf{w}^2\}, \{\mathbf{w}^3\}\big\}$$

We say a partition profile Π is *consistent* with a matching μ if for every j and every $\mathbf{w}'$, $\mathbf{w}''(\mu^{-1}(j)) = \mathbf{w}'(\mu^{-1}(j))$ for all $\mathbf{w}'' \in \Pi_j(\mathbf{w}')$. Indeed, as firm j can observe the type of her current employee $\mu^{-1}(j)$, the type of $\mu^{-1}(j)$ must be equal to $\mathbf{w}'(\mu^{-1}(j))$ for each type assignment $\mathbf{w}''$ considered possible by firm j when $\mathbf{w}'$ is the true type assignment. A *state* of the matching market, $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, specifies an allocation $(\mu, \mathbf{p})$, a type-assignment $\mathbf{w}$ and a partition profile Π that is consistent with μ. The information partition profile in Example 1 is the coarsest partition profile that satisfies consistency. Alternatively, we allow for any finer partition profile that incorporates firms' information that is exogenously given. For instance, given the allocation $(\mu, \mathbf{p})$ in Example 1 and the true type-assignment $\mathbf{w}^1$, the following three partition profiles are all consistent with μ.

Profile 1	Profile 2	Profile 3
$\Pi_a = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2\}, \{\mathbf{w}^3\}\}$	$\Pi_a = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2, \mathbf{w}^3\}\}$	$\Pi_a = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2\}, \{\mathbf{w}^3\}\}$
$\Pi_b = \{\{\mathbf{w}^1, \mathbf{w}^2\}, \{\mathbf{w}^3\}\}$	$\Pi_b = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2\}, \{\mathbf{w}^3\}\}$	$\Pi_b = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2\}, \{\mathbf{w}^3\}\}$

3 Stability with Incomplete Information

3.1 Individual Rationality

A state is said to be individually rational if each agent receives at least the payoff of remaining unmatched, which is zero.

Definition 1. *A state* $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is said to be* ***individually rational*** *if*

$$\nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \geq 0 \text{ for all } i \in I \text{ and}$$
$$\phi_{\mathbf{w}(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \geq 0 \text{ for all } j \in J.$$

3.2 Blocking

The notion of incomplete-information "blocking" naturally extends its complete-information counterpart. In particular, a matching is blocked if some worker-firm pair (i, j), where i and j are not matched with each other, can mutually benefit from being matched with each other. We propose the following definition of "blocking" which extends LMPS's notion to accommodate private information of the firms.

Definition 2. *A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be* ***blocked*** *if there exists a worker-firm pair (i, j) and a payment $p \in \mathbb{R}$ such that i is not matched with j under μ, i prefers to be rematched with j at p to his current match, and j prefers to be rematched with i at p as long as i would switch at that price, i.e.,*

$$\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p \quad > \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ and} \tag{2}$$

$$\phi_{\mathbf{w}'(i),\mathbf{f}(j)} - p \quad > \phi_{\mathbf{w}'(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \tag{3}$$

for all $\mathbf{w}' \in \Pi_j(\mathbf{w})$ satisfying

$$\nu_{\mathbf{w}'(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}. \tag{4}$$

We call the pair (i, j) a *blocking pair*, and the tuple $(i, j; p)$ a *blocking combination*, for the state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ when conditions (2)-(4) are satisfied. For firm j to participate a potential blocking pair at state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, it must guarantee an improvement for every relevant type assignment. More precisely, when a firm j considers forming a potential blocking pair (i, j) with worker i, at some potential salary p, a type assignment $\mathbf{w}'$ is relevant for firm j when $\mathbf{w}' \in \Pi_j(\mathbf{w})$ and (4) holds. All type assignments violating (4) are irrelevant due to the worker's objection.

The following fact is an immediate consequence of the worst-case desideratum in Definition 2. It says that a blocking is more likely to exist if agents have more precise information about the workers.[12]

Fact 1. *Suppose that Π' is a finer partition profile than Π. If state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is blocked, then state $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ is also blocked.*

3.3 Stability

When information is complete, stable matchings embodies the intuition that when "the agents have a very good idea of one another's preferences and have easy access to each other,..., we might expect that stable matchings will be especially likely to occur" (Roth and Sotomayor, 1990, pp. 22).[13] In this case, a stable state is simply a state that is individually rational and not blocked.

In contrast, with incomplete information, we argue that individual rationality and the absence of blocking pair are no longer sufficient to describe a "stable state." To be precise, the partition Π_j describes only firm j's imprecise idea about the workers' information. However, the absence of blocking pairs may still provide further information to agents.

[12] A partition profile Π' is said to be *finer* than another partition profile Π if Π'_k is finer than Π_k for all $k \in I \cup J$.

[13] Information is *complete* if every agent know the true type-assignment, whatever it is. In our notation, this is to say that $\Pi_j(\mathbf{w}') = \{\mathbf{w}'\}$ for all $j \in J$ and $\mathbf{w}' \in \Omega$.

Once the firms' information partitions become finer, the worst case improves and hence new blocking pairs may arise. This is illustrated in the following example.[14]

Example 2. *Consider a job market which is the same as in Example 1, except for the partition profile. To be precise, $I = \{\alpha, \beta\}$ and $J = \{a, b\}$. The matching μ is given by $\mu(\alpha) = a$ and $\mu(\beta) = b$, and the payment scheme is given by $\mathbf{p}_{\alpha,a} = \mathbf{p}_{\beta,b} = 0$. The information structure is given by $\Omega = \{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\}$ and $\mathbf{w}^* = \mathbf{w}^1$. The details of Ω is given in (1) and the partition profile is given below.*

$$\begin{aligned}\Pi_a &= \big\{\{\mathbf{w}^1\}, \{\mathbf{w}^2\}, \{\mathbf{w}^3\}\big\}\\ \Pi_b &= \big\{\{\mathbf{w}^1, \mathbf{w}^2\}, \{\mathbf{w}^3\}\big\}\end{aligned}$$

Denote firm a's type and firm b's type by f_a and f_b, respectively. We specify the premuneration values under $\mathbf{w}^1$ and $\mathbf{w}^2$ as below ($\mathbf{w}^3$ is irrelevant for the discussion).

$$\mathbf{w}^1: \quad \begin{array}{cccc} \boxed{\nu_{w_\alpha,f_a} = 4} & \nu_{w_\beta,f_a} = 4 & \underline{\nu_{w_\alpha,f_b} = 5} & \nu_{w_\beta,f_b} = 5 \\ \phi_{w_\alpha,f_a} = 4 & \phi_{w_\beta,f_a} = 4 & \underline{\phi_{w_\alpha,f_b} = 5} & \boxed{\phi_{w_\beta,f_b} = 5} \end{array}$$

$$\mathbf{w}^2: \quad \begin{array}{cccc} \nu_{w'_\alpha,f_a} = 2 & \underline{\nu_{w_\beta,f_a} = 4} & \nu_{w'_\alpha,f_b} = 3 & \boxed{\nu_{w_\beta,f_b} = 5} \\ \boxed{\phi_{w'_\alpha,f_a} = 2} & \underline{\phi_{w_\beta,f_a} = 4} & \phi_{w'_\alpha,f_b} = 3 & \phi_{w_\beta,f_b} = 5 \end{array}$$

The numbers are designed to satisfy the following properties:

(i) $\nu_{w_\alpha,f_a} + \phi_{w_\beta,f_b} = 4+5 < 5+5 = \nu_{w_\alpha,f_b} + \phi_{w_\alpha,f_b}$. *Therefore, with complete information (the first two rows), (α, b) is a blocking pair. Moreover, (β, a) is not a blocking pair.*

(ii) $\nu_{w'_\alpha,f_a} + \phi_{w_\beta,f_b} = 2+5 < 4+4 = \nu_{w_\beta,f_a} + \phi_{w_\beta,f_a}$. *Therefore,* were $\mathbf{w}^2$ *the true type-assignment (the intermediate two rows), (β, a) would be a blocking pair. Moreover,* were $\mathbf{w}^2$ *the true type-assignment, (α, b) would not be a blocking pair.*

(iii) $\nu_{w'_\alpha,f_b} - \nu_{w'_\alpha,f_a} = 3-2 = 5-4 = \nu_{w_\alpha,f_b} - \nu_{w_\alpha,f_a}$. *Therefore, firm b cannot discriminate two types w_α and w'_α of worker α by designing a rematching salary p.*

Obviously, the state $(\mu, \mathbf{p}, \mathbf{w}^1, \Pi)$ is individually rational. Moreover, $(\mu, \mathbf{p}, \mathbf{w}^1, \Pi)$ is not blocked because of (i) and (iii).

Now we investigate what agents can learn from the fact that $(\mu, \mathbf{p}, \mathbf{w}^1, \Pi)$ is not blocked. First, by (ii), we know that $(\mu, \mathbf{p}, \mathbf{w}^2, \Pi)$ is blocked. Therefore, knowing the fact of no rematching, firm b would know that $\mathbf{w}^2$ should not be the true type-assignment; otherwise there should be a rematching. Hence, taking into account the inference from the lack of rematching, firm b's updated partition is

$$\Pi'_b := \big\{\{\mathbf{w}^1\}, \{\mathbf{w}^2\}, \{\mathbf{w}^3\}\big\}.$$

[14] The idea of Example 2 is inherited from Subsection 2.2.2 of LMPS.

One can easily check that (α, b) is a blocking pair for the state $(\mu, \mathbf{p}, \mathbf{w}^1, \Pi')$, where $\Pi'_a := \Pi_a$ for notation convenience. For example, a rematching between worker α and firm b at a salary of -0.5 makes both of them better off.

The "stability" notion that we propose below captures, on top of individual rationality and no blocking, the requirement that individual rationality and no blocking pair provide no further information to agents. This additional "information stability" requirement is specific to the incomplete-information environment.

To formulate the information stability requirement, we recall that $(\mu, \mathbf{p}, \Pi)$ is publicly observed. We then partition Ω into two subsets, depending on whether state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is individually rational and not blocked or not. More precisely, we define the binary partition $N^{(\mu,\mathbf{p},\Pi)}$ such that for any two type assignments $\mathbf{w}'$ and $\mathbf{w}''$, $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}') = N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}'')$ if and only if one of the following conditions hold:

(i) Both $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ are either not individually rational or blocked;

(ii) Both $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ are individually rational and not blocked.

Then the information contained in "the fact of individual rationality and no blocking pair" (and/or its compliment "the fact of not being individual rational or being blocked") is summarized in $N^{(\mu,\mathbf{p},\Pi)}$.

Next, we describe how to aggregate two pieces of information which are represented by partitions. In short, the aggregated information is represented by the join of the two partitions.[15] Consider a firm j who has two pieces of information Π_j and $N^{(\mu,\mathbf{p},\Pi)}$. At each type-assignment $\mathbf{w}'$, deemed to be true, firm j knows that the true type-assignment $\mathbf{w}'$ lies in the set $\Pi_j(\mathbf{w}')$, and that the true type-assignment $\mathbf{w}'$ lies in the set $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}')$. Therefore, firm j knows that the true type-assignment $\mathbf{w}'$ lies in the set $\Pi_j(\mathbf{w}') \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}')$.

To sum up, the information provided by "the fact of individual rationality and no blocking pair" leads to a new partition profile $H_{\mu,\mathbf{p}}(\Pi)$ defined as

$$[H_{\mu,\mathbf{p}}(\Pi)]_j(\mathbf{w}') := \Pi_j(\mathbf{w}') \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{ and all } j \in J. \tag{5}$$

If $H_{\mu,\mathbf{p}}(\Pi) \neq \Pi$, then "the fact of individual rationality and no blocking pair" do provide some further information to some agents.

A state is said to be stable if it is individually rational, not blocked, and no information can be inferred from the fact that the state is individually rational and not blocked.

Definition 3. *A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be **stable** if*

1. *it is individually rational,*
2. *it is not blocked, and*
3. *Π is a fixed point of $H_{\mu,\mathbf{p}}$, i.e., $H_{\mu,\mathbf{p}}(\Pi) = \Pi$.*

[15] The *join* of two partitions is the coarsest common refinement of them. See Aumann (1976).

It is well known that in the current setup, a stable matching exists if information is complete, i.e., if $\Pi_j(\mathbf{w}') = \{\mathbf{w}'\}$ for all $j \in J$ and $\mathbf{w}' \in \Omega$, then for each $\mathbf{w}$ there exists an allocation $(\mu, \mathbf{p})$ such that the state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable (see Shapley and Shubik (1971) and Crawford and Knoer (1981)). The existence of a stable state is guaranteed by the existence of a complete-information stable state, in which the particular partition Π is a fixed point of any $H_{\mu,\mathbf{p}}$ that is induced by any binary partition. Even with discrete transfer, stable states are well defined and the existence is still guaranteed (see Theorem 1 of Crawford and Knoer (1981)).

Remark 1. *Conceptually, the information partition profile Π could be viewed as both an input variable and an output variable. In the former case, we could ask whether or not an allocation $(\mu, \mathbf{p})$ is stable at the given information structure $(\mathbf{w}, \Pi)$. In the latter case we ask analogously whether or not a state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable. This flexibility makes the partition-profile representation crucial for the current paper since Π will be endogenized once we study matching processes, where Π is both an input and an output.*

3.4 Equivalence of Two Stability Notions

In this subsection, we compare Definition 3 and the stability notion introduced by LMPS. The stability notion of LMPS is *ex ante* in that it is independent of the true type-assignment. One can imagine that there is an outside analyst who knows the model except for the true type-assignment, and who wants to predict possible outcomes for the market. As usual, stable outcomes are individually rational and immune to blocking pairs. Formally, an *(ex post) matching outcome* $(\mu, \mathbf{p}, \mathbf{w})$ specifies an allocation and a type-assignment. The individual rationality of a matching outcome is defined in a same manner as in Definition 1, i.e., each agent has a nonnegative payoff. The blocking notion of LMPS is designed to exclude only outcomes that the analyst can be certain are "blocked."

Definition 4. *(LMPS) Let Σ be a nonempty subset of individually rational matching outcomes. A matching outcome $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma$ is Σ-**blocked** if there exists a worker-firm pair (i, j) and a payment $p \in \mathbb{R}$ that satisfy*

$$\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \quad \textit{and} \tag{6}$$

$$\phi_{\mathbf{w}'(i),\mathbf{f}(j)} - p > \phi_{\mathbf{w}'(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \tag{7}$$

for all $\mathbf{w}' \in \Omega$ satisfying

$$(\mu, \mathbf{p}, \mathbf{w}') \in \Sigma \tag{8}$$

$$\mathbf{w}'(\mu^{-1}(j)) = \mathbf{w}(\mu^{-1}(j)) \tag{9}$$

$$\nu_{\mathbf{w}'(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}. \tag{10}$$

*A matching outcome $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma$ is Σ-**stable** if it is not Σ-blocked.*

Condition (6) says that worker i prefers firm j at salary p to his current match. Conditions (8-10) mean that firm j considers only "reasonable" type-assignments, which are consistent with (i) the outcome set Σ, (ii) its "observation" $\mathbf{w}(\mu^{-1}(j))$ and (iii) worker i's willingness to block the outcome $(\mu, \mathbf{p}, \mathbf{w})$ with j at p. Condition (7) says that under *any* "reasonable" type-assignments, firm j prefers worker i at p to its current match. Intuitively, blocking conditions in Definition 4 say that if $\mathbf{w}$ *were* the true type-assignment, then $(\mu, \mathbf{p}, \mathbf{w})$ would be blocked based on the information of Σ, i.e., only outcomes in Σ are possible.

The set of outcomes that are immune to the blocking described in Definition 4 is given by the iteration process below. Let Σ^0 be the set of all individually rational outcomes. For $k \geq 1$, define

$$\Sigma^k := \left\{(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^{k-1} : (\mu, \mathbf{p}, \mathbf{w}) \text{ is } \Sigma^{k-1} - \textit{stable}\right\}. \tag{11}$$

The set of *incomplete-information stable outcomes* in LMPS is given by $\Sigma^\infty := \cap_{k=1}^\infty \Sigma^k$.

The following theorem establishes the equivalence between our stability notion and that of LMPS. On the one hand, as long as $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$, we can find at least one (possibly many) partition profile Π such that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is a stable state. That is, each incomplete-information stable outcome can be supported as a part of some stable market state. On the other hand, as long as we can find one partition profile Π to support $(\mu, \mathbf{p}, \mathbf{w})$, the outcome $(\mu, \mathbf{p}, \mathbf{w})$ must be stable in the sense of LMPS.

Theorem 1. $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$ *if and only if there exists a partition profile* Π *such that* $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is stable.*

We prove the necessity part by constructing Π. To be precise, given $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$, we denote the partition over Ω that indicates the type of worker i by $O^{\{i\}}$, i.e.,

$$O^{\{i\}}(\mathbf{w}') := \{\mathbf{w}'' \in \Omega : \mathbf{w}''(i) = \mathbf{w}'(i)\}. \tag{12}$$

Define $\bar{\Pi}^0_j(\mathbf{w}') := O^{\{\mu^{-1}(j)\}}(\mathbf{w}')$ for all $\mathbf{w}' \in \Omega$ and all $j \in J$ as the partition profile induced from Ω when firms only know the type of their own employees. The desired Π is constructed by the following iteration starting with $\bar{\Pi}^0$ and, for $k = 1, 2, \ldots,$

$$\bar{\Pi}^k := H_{\mu,\mathbf{p}}(\bar{\Pi}^{k-1}). \tag{13}$$

We first show that for any $k = 1, 2 \ldots,$ the intermediate state $(\mu, \mathbf{p}, \mathbf{w}, \bar{\Pi}^k)$ is not blocked. Next, by construction, $\bar{\Pi}^\infty$ is a fixed point of $H_{\mu,\mathbf{p}}$. Therefore, by Definition 3 the state $(\mu, \mathbf{p}, \mathbf{w}, \bar{\Pi}^\infty)$ constructed by (13) is a stable state. The sufficiency part is proved by applying Proposition 2 of LMPS, where we construct a *self-stabilizing set.* A nonempty set of individually rational matching outcomes E is *self-stabilizing* if every $(\mu, \mathbf{p}, \mathbf{w}) \in E$ is E-stable. See (Liu et al., 2014, pp. 555).

Proof of Theorem 1. **Necessity.** Suppose $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$. We proceed to construct a partition profile Π such that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is a stable state. Before that, we derive the implications of $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$.

We first define the partition profile $\hat{\Pi}$ induced by Σ^∞. Denote the binary partition over Ω that indicates whether $(\mu, \mathbf{p}, \mathbf{w}') \in \Sigma^\infty$ or not by $T_{\mu,\mathbf{p}}$, i.e., $T_{\mu,\mathbf{p}}(\mathbf{w}') = T_{\mu,\mathbf{p}}(\mathbf{w}'')$ if and only if one of the following conditions hold:

(i) $(\mu, \mathbf{p}, \mathbf{w}') \in \Sigma^\infty$ and $(\mu, \mathbf{p}, \mathbf{w}'') \in \Sigma^\infty$;

(ii) $(\mu, \mathbf{p}, \mathbf{w}') \notin \Sigma^\infty$ and $(\mu, \mathbf{p}, \mathbf{w}'') \notin \Sigma^\infty$.

Each firm j has two pieces of information: the type of its own employee; and the fact that only outcomes in $\Sigma^\infty_{\mu,\mathbf{p}}$ are possible. Formally, this means:

$$\hat{\Pi}_j(\mathbf{w}') := O^{\{\mu^{-1}(j)\}}(\mathbf{w}') \cap T_{\mu,\mathbf{p}}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{ and all } j \in J.$$

Next, since $(\mu, \mathbf{p}, \mathbf{w})$ is not Σ^∞-blocked, by checking Definition 3 and 4 we know that $(\mu, \mathbf{p}, \mathbf{w}, \hat{\Pi})$ is not blocked. Similarly, $(\mu, \mathbf{p}, \mathbf{w}', \hat{\Pi})$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$.

Now we construct the desired partition profile Π. Consider a particular partition profile $\bar{\Pi}^0$ defined as

$$\bar{\Pi}^0_j(\mathbf{w}') := O^{\{\mu^{-1}(j)\}}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{ and all } j \in J.$$

Then $(\mu, \mathbf{p}, \mathbf{w}, \bar{\Pi}^0)$ is not blocked by Fact 1 (otherwise $(\mu, \mathbf{p}, \mathbf{w}, \hat{\Pi})$ is blocked since $\hat{\Pi}$ is finer than $\bar{\Pi}^0$). Similarly, $(\mu, \mathbf{p}, \mathbf{w}', \bar{\Pi}^0)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$. Therefore, $N^{(\mu,\mathbf{p},\bar{\Pi}^0)}(\mathbf{w}) \supset T_{\mu,\mathbf{p}}(\mathbf{w})$. As a result, adding the information $N^{(\mu,\mathbf{p},\bar{\Pi}^0)}$ does not change the consideration sets of firms, i.e.,

$$[H_{\mu,\mathbf{p}}(\bar{\Pi}^0)]_j(\mathbf{w}') = \bar{\Pi}^0_j(\mathbf{w}') \text{ for all } \mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w}) \text{ and all } j \in J.$$

Since $(\mu, \mathbf{p}, \mathbf{w}', \bar{\Pi}^0)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$, we know that $(\mu, \mathbf{p}, \mathbf{w}', H_{\mu,\mathbf{p}}(\bar{\Pi}^0))$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$, which implies that $N^{(\mu,\mathbf{p},H_{\mu,\mathbf{p}}(\bar{\Pi}^0))}(\mathbf{w}) \supset T_{\mu,\mathbf{p}}(\mathbf{w})$. Keep applying this argument and define

$$\bar{\Pi}^k := H_{\mu,\mathbf{p}}(\bar{\Pi}^{k-1}) \text{ for } k = 1, 2, \dots,$$

until we find a fixed point of $H_{\mu,\mathbf{p}}$. This must be done within finitely many times because the partition profile gets finer whenever it is not a fixed point. Denote the fixed point by $\bar{\Pi}^\infty$. Then induction shows that $N^{(\mu,\mathbf{p},\bar{\Pi}^\infty)}(\mathbf{w}) \supset T_{\mu,\mathbf{p}}(\mathbf{w})$, and that $(\mu, \mathbf{p}, \mathbf{w}', \bar{\Pi}^\infty)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$. Particularly for $\mathbf{w}' = \mathbf{w}$, we know that $(\mu, \mathbf{p}, \mathbf{w}, \bar{\Pi}^\infty)$ is not blocked. Therefore, we conclude that $\Pi := \bar{\Pi}^\infty$ is the desired partition profile.

Sufficiency. Suppose there exists a partition profile Π such that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable.

We proceed to construct a self-stabilizing set. Let

$$\Sigma^N := \left\{(\mu, \mathbf{p}, \mathbf{w}') : \mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})\right\}.$$

Then obviously $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^N$. We claim that $(\mu, \mathbf{p}, \mathbf{w}')$ is not Σ^N-blocked for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$. Then, Σ^N is a self-stabilizing set. By Proposition 2 of LMPS, we have

$$(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^N \subset \Sigma^\infty.$$

To see the claim, we first note that the stability of $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ implies $H_{\mu,\mathbf{p}}(\Pi) = \Pi$. This in turn implies that for each j and each $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$, we have

$$\Pi_j(\mathbf{w}') \subseteq N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}).$$

Furthermore, since $\mathbf{w}''(\mu^{-1}(j)) = \mathbf{w}(\mu^{-1}(j))$ for all $\mathbf{w}'' \in \Pi_j(\mathbf{w}')$, we know that

$$\Pi_j(\mathbf{w}') \subseteq \left\{\mathbf{w}'' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}) : \mathbf{w}''(\mu^{-1}(j)) = \mathbf{w}(\mu^{-1}(j))\right\}. \tag{14}$$

Suppose to the contrary that $(\mu, \mathbf{p}, \mathbf{w}')$ is not Σ^N-blocked. Then $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ is blocked by (14) and Fact 1, a contradiction to $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$. Therefore, the claim holds. This completes the proof. □

3.5 Stable Allocations

Given a type-assignment $\mathbf{w}$, the payoffs of agents are actually determined by the allocation. In this subsection, we concentrate on the set of allocations that can arise in stable states, which is a set of allocations that are stable under some partition profile. Denote this set of allocations by $\mathscr{S}(\mathbf{w})$, i.e.,

$$\mathscr{S}(\mathbf{w}) := \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \text{ a partition profile } \Pi \text{ s.t. } (\mu, \mathbf{p}, \mathbf{w}, \Pi) \text{ is stable}\}. \tag{15}$$

The ex ante set of stable allocations is then defined as $\mathscr{S} := \bigcup_{\mathbf{w}\in\Omega} \mathscr{S}(\mathbf{w})$. By the definition of $\mathscr{S}(\mathbf{w})$, we have

$$\begin{aligned}\mathscr{S} =& \bigcup_{\mathbf{w}\in\Omega} \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \text{ a partition profile } \Pi \text{ such that } (\mu, \mathbf{p}, \mathbf{w}, \Pi) \text{ is stable } \}\\ =&\{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \mathbf{w} \in \Omega \text{ and a partition profile } \Pi \text{ such that } (\mu, \mathbf{p}, \mathbf{w}, \Pi) \text{ is stable } \}.\end{aligned}$$

The following corollary, an immediate result of Theorem 1, says that our notion of stability (Definition 1-3) is consistent with that of LMPS in terms of allocation. In other words, if we extract all allocations from the set of incomplete-information stable outcomes (defined by LMPS), then the result is exactly $\mathscr{S}$.

Corollary 1. $\mathscr{S} = \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \mathbf{w} \in \Omega \ s.t. \ (\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty\}$.

In addition, when we extract from Σ^∞ all allocations associated with a particular $\mathbf{w}$, the result is exactly the same as $\mathscr{S}(\mathbf{w})$.[16]

Corollary 2. $\mathscr{S}(\mathbf{w}) = \{(\mu, \mathbf{p}) \in \mathscr{A} : (\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty\}$.

4 Matching Processes with Incomplete Information

Consider a job market in which any worker-firm pair can freely choose to be matched to each other, and any agent can freely opt to be unmatched. Suppose that agents are *myopic, i.e., once an agent or a worker-firm pair finds an opportunity to improve their status quo, they will do so by either remaining unmatched or finding a new partner.* These individual and/or pairwise rematchings lead to a sequence of market states, which is referred as a matching process.

In this section, we study whether the stable states we defined can be seen as the outcome of a matching process. In particular, we show that with probability one an arbitrary, random Learning-Blocking Path converges to an incomplete-information stable state after finitely many rematchings. Throughout this section, we will fix a realized type-assignment $\mathbf{w}^*$.

4.1 learning-blocking paths

With incomplete information, a matching process is usually associated with a *learning process* (which corresponds to a sequence of refinements in the partitional information of the agents).

In this subsection, we formally specify how agents update their information according to new observation. Given a state $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ whose the partition profile is common knowledge, each firm j may observe one of the following two situations:

1. there is no rematching; or
2. there is a rematching of some blocking combination $(i, j; p)$.

In case 1, it is commonly known among the firms that $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is not blocked, a event which can be distinguished from the event of the state being blocked. As a result, they update their information according to the partition $N^{(\mu,\mathbf{p},\Pi)}$ defined in subsection

[16]We can easily construct a matching market such that $\mathscr{S}(\mathbf{w}) \subsetneq \mathscr{S}$ for some/all $\mathbf{w} \in \Omega$. To see this, consider a market with one firm and two workers, α and β. There are two possible type-assignments, $\mathbf{w}$ and $\mathbf{w}'$. Suppose both possible matches are individually rational for some payments. Suppose additionally that $\mathbf{w}$ and $\mathbf{w}'$ never agree on any worker's type. Hence, any match will give the firm full information. Under $\mathbf{w}$, the firm strictly prefers α to β. Thus, only a match between α and the firm could be part of the stable states. Under $\mathbf{w}'$, the firm strictly prefers β to α. Thus, only a match between β and the firm could be part of the stable states. Therefore, we have both $\mathscr{S}(\mathbf{w}) \subsetneq \mathscr{S}$ and $\mathscr{S}(\mathbf{w}') \subsetneq \mathscr{S}$.

3.3, i.e., the information structure now becomes $H_{\mu,\mathbf{p}}(\Pi)$ defined by (5). This case has been illustrated by Example 2.

In case 2, it is commonly known that all firms observe the rematching. Additionally, they can distinguish two events: one which permits $(i, j; p)$ as a blocking combination and one which does not. Denote the partition identifying the blocking combination $(i, j; p)$ by $B^{(\mu,\mathbf{p},\Pi;i,j;p)}$, i.e., $B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}') = B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}'')$ if and only if one of the following conditions hold:

(i) $(i, j; p)$ blocks both $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$;

(ii) $(i, j; p)$ neither a blocking combination for $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$, nor for $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$.

Moreover, it is common knowledge that firm j has observed the type of worker i. That is, firm j gets an extra piece of information represented by $O^{\{i\}}$, which is defined by (12).

We say that a state $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ is *derived from another state* $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ *by satisfying a blocking combination* $(i, j; p)$ *for* $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$, which is denoted by $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi') \xleftarrow{(i,j;p)} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$, if

$$\begin{gathered}
\mu'(i) = j; \qquad \mu'((\mu)^{-1}(j)) = \emptyset; \\
\mu'(i') = \mu(i') \text{ for all } i' \in I \text{ s.t. } i' \neq (\mu)^{-1}(j) \text{ and } i' \neq i; \\
\mathbf{p}'_{i,j} = p; \qquad \mathbf{p}'_{(\mu)^{-1}(j),\emptyset} = 0; \\
\mathbf{p}'_{i',\mu'(i')} = \mathbf{p}_{i',\mu(i')} \text{ for all } i' \in I \text{ s.t. } i' \neq (\mu)^{-1}(j) \text{ and } i' \neq i,
\end{gathered}$$

and

$$\Pi'_j(\mathbf{w}) = \Pi_j(\mathbf{w}) \cap O^{\{i\}}(\mathbf{w}) \text{ for all } \mathbf{w} \in \Omega; \tag{16}$$

$$\Pi'_{j'}(\mathbf{w}) = \Pi_{j'}(\mathbf{w}) \cap B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}) \text{ for all } \mathbf{w} \in \Omega \text{ and all } j' \neq j. \tag{17}$$

In other words, $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ is derived from another state $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ by satisfying a blocking combination $(i, j; p)$ for $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ if all of the following conditions are met:[17]

(i) worker i and firm j rematch at the salary p;

(ii) the previous partners of i and j, if any, become unmatched;

(iii) every firm updates its information according to the public observation of the rematching; and

(iv) firm j updates its information according to the additional observation of worker i's true type.

[17] Technically, we allow one agent of i and j to be $\emptyset$, in which case $p = 0$. In particular, $i = \emptyset$ means that firm j dismisses its employee $\mu^{-1}(j)$; $j = \emptyset$ means that worker i resigns from his firm $\mu(i)$. Thus, the operation $\xleftarrow{(i,j;p)}$ and the term "rematching" apply to both pairs and individuals.

In Example 3 below, a blocking combination is satisfied with information updating.

Example 3. *Consider a job market which is the same as in Example 1, except for the true type-assignment. To be precise, $I = \{\alpha, \beta\}$, $J = \{a, b\}$, and $\Omega = \{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\}$, where $\mathbf{w}^* = \mathbf{w}^2$. The matching μ is given by $\mu(\alpha) = a$ and $\mu(\beta) = b$, and the payment scheme is given by $\mathbf{p}_{\alpha,a} = \mathbf{p}_{\beta,b} = 0$. The details of Ω is given in (1) and the partition profile is given below.*

$$\Pi_a = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2, \mathbf{w}^3\}\}$$
$$\Pi_b = \{\{\mathbf{w}^1, \mathbf{w}^2\}, \{\mathbf{w}^3\}\}$$

Denote firms' a's type and firm b's type by f_a and f_b, respectively. We specify the premuneration values as below.

$$\begin{array}{lcccc}
\mathbf{w}^1: & \boxed{\nu_{w_\alpha,f_a} = 4} & \nu_{w_\beta,f_a} = 4 & \underline{\nu_{w_\alpha,f_b} = 5} & \nu_{w_\beta,f_b} = 5 \\
 & \phi_{w_\alpha,f_a} = 4 & \phi_{w_\beta,f_a} = 4 & \underline{\phi_{w_\alpha,f_b} = 5} & \boxed{\phi_{w_\beta,f_b} = 5} \\
 & & & & \\
\mathbf{w}^2: & \nu_{w'_\alpha,f_a} = 2 & \underline{\nu_{w_\beta,f_a} = 4} & \nu_{w'_\alpha,f_b} = 3 & \boxed{\nu_{w_\beta,f_b} = 5} \\
 & \boxed{\phi_{w'_\alpha,f_a} = 2} & \underline{\phi_{w_\beta,f_a} = 4} & \phi_{w'_\alpha,f_b} = 3 & \phi_{w_\beta,f_b} = 5 \\
 & & & & \\
\mathbf{w}^3: & \nu_{w'_\alpha,f_a} = 2 & \underline{\nu_{w'_\beta,f_a} = 5} & \nu_{w'_\alpha,f_b} = 3 & \boxed{\nu_{w'_\beta,f_b} = 4} \\
 & \boxed{\phi_{w'_\alpha,f_a} = 2} & \underline{\phi_{w'_\beta,f_a} = 5} & \phi_{w'_\alpha,f_b} = 3 & \phi_{w'_\beta,f_b} = 4
\end{array}$$

The numbers are designed to satisfy the following properties:

(i) $\nu_{w_\alpha,f_a} + \phi_{w_\beta,f_b} = 4 + 5 < 5 + 5 = \nu_{w_\alpha,f_b} + \phi_{w_\alpha,f_b}$. *Therefore,* were $\mathbf{w}^1$ *the true type-assignment (the first two rows), (α, b) would be a blocking pair. Moreover,* were $\mathbf{w}^1$ *the true type-assignment (β, a) would not be a blocking pair.*

(ii) $\nu_{w'_\alpha,f_a} + \phi_{w_\beta,f_b} = 2+5 < 4+4 = \nu_{w_\beta,f_a} + \phi_{w_\beta,f_a}$. *Therefore, with complete information (the second two rows), $(\beta, a)$is a blocking pair. Moreover, (α, b) is not a blocking pair.*

(iii) Similarly, $\nu_{w'_\alpha,f_a} + \phi_{w_\beta,f_b} = 2 + 5 < 5 + 5 = \nu_{w'_\beta,f_a} + \phi_{w'_\beta,f_a}$. *Therefore,* were $\mathbf{w}^3$ *the true type-assignment (the last two rows), (β, a) would be a blocking pair. Moreover,* were $\mathbf{w}^3$ *the true type-assignment, (α, b) would not be a blocking pair.*

Obviously, the state $(\mu, \mathbf{p}, \mathbf{w}^2, \Pi)$ is blocked by, say, $(\beta, a, 1.5)$ because of (ii) and (iii). Note that

$$O^{\{\beta\}} = \{\{\mathbf{w}^1, \mathbf{w}^2\}, \{\mathbf{w}^3\}\}$$

and that by (i)-(iii),

$$B^{(\mu,\mathbf{p},\Pi;\beta,a;1.5)} = \{\{\mathbf{w}^1\}, \{\mathbf{w}^2, \mathbf{w}^3\}\}.$$

Then the new allocation is given by $\mu'(\beta) = a$ with $\mathbf{p}'_{\beta,a} = 1.5$ and $\mu'(\alpha) = \emptyset$; the updated *partition profile incorporating $O^{\{\beta\}}$ and $B^{(\mu,\mathbf{p},\Pi;\beta,a,1.5)}$, as in (16)-(17), is given as*

$$\Pi'_a = \{\{\mathbf{w}^1\},\{\mathbf{w}^2\},\{\mathbf{w}^3\}\}$$
$$\Pi'_b = \{\{\mathbf{w}^1\},\{\mathbf{w}^2\},\{\mathbf{w}^3\}\}\,.$$

We describe a *learning-blocking path* as a sequence of states $\{(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)\}_{l=0}^{L}$ such that for any two adjacent states $(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)$ and $(\mu^{l+1},\mathbf{p}^{l+1},\mathbf{w}^*,\Pi^{l+1})$,

(i) if $(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)$ is not blocked, then $(\mu^{l+1},\mathbf{p}^{l+1}) = (\mu^l,\mathbf{p}^l)$ and $\Pi^{l+1} = H_{\mu^l,\mathbf{p}^l}(\Pi^l)$;

(ii) if $(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)$ is blocked, then $(\mu^{l+1},\mathbf{p}^{l+1},\mathbf{w}^*,\Pi^{l+1}) \xleftarrow{(i,j;p)} (\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)$, where $(i,j;p)$ is a blocking combination for $(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)$.

A learning-blocking path is said to be *finite* if $L<\infty$. A learning-blocking path $\{(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)\}_{l=0}^{L}$ is said to *converge within finitely many steps* if there exists a finite $T<L$ such that $(\mu^{T+1},\mathbf{p}^{T+1},\mathbf{w}^*,\Pi^{T+1}) = (\mu^T,\mathbf{p}^T,\mathbf{w}^*,\Pi^T)$.

A learning-blocking path may not converge because of cycles, as in the example provided by Knuth (1976) in an ordinal-preference setting.[18] Intuitively, when a rematching happens, the rematched worker and firm both get better off while their previous partners become unmatched, i.e., worse off. Agents who got worse off are easier than before to find blocking opportunities since any positive payoff would be an improvement. New blocking opportunities may, in turn, drag down the payoffs of the previously improved agents. As a result, there may be cycles along a learning-blocking path.

We close this subsection by noticing that the learning-blocking paths include the complete-information blocking paths as special cases. To be precise, if $\Pi_j(\mathbf{w}) = \{\mathbf{w}\}$ for all $\mathbf{w}\in\Omega$ and $j\in J$, we are back to the complete-information environment. In this case, the partition profile is already the finest, which implies that no extra information can be obtained from any observation. Thus, a learning-blocking path is simply a blocking path discussed in the literature, i.e., a sequence of allocations where each allocation is derived from its immediate predecessor by satisfying one of the blocking combinations for the predecessor (e.g. RV and Chen et al. (2010)).

4.2 Convergence of learning-blocking paths

As we discussed in the last subsection, starting at an initial state $(\mu^0,\mathbf{p}^0,\mathbf{w}^*,\Pi^0)$, an arbitrary learning-blocking path $\{(\mu^l,\mathbf{p}^l,\mathbf{w}^*,\Pi^l)\}_{l=0}^{L}$ may not converge because of cycles. However, learning-blocking paths are not completely chaotic. Given the initial state $(\mu^0,\mathbf{p}^0,\mathbf{w}^*,\Pi^0)$, we first show a deterministic convergence result. That is, given an arbitrary initial state, we can carefully choose blocking combinations when there are many, such that the constructed learning-blocking path must converge to a stable state within finitely many steps (Theorem 2). Then we show a random convergence result. That

[18] One can easily construct an example with cycles in the transferable-utility setting.

is, given an arbitrary initial state, the learning-blocking path resulted from randomly satisfying blocking combinations (when there are many) must almost surely converge to a stable state within finitely many steps (Theorem $2'$). The former convergence concerns whether there exists *one* learning-blocking path that leads to a stable state; while the latter concerns the probability of reaching a stable state when blocking combinations are randomly satisfied.

4.2.1 Deterministic Path to Stability

Before stating our results, we need the following assumption.

Assumption 1. *Payments permitted in the job market are integers.*[19]

Indeed, payments in practice are measured in monetary units and hence integers.[20]

Given an arbitrary initial state, we show in the following theorem that by carefully choosing blocking pairs at each state, we can construct a finite learning-blocking path that ends with a stable state.

Theorem 2. *Suppose Assumption 1 holds. Then starting from any arbitrary initial state, there exists a finite learning-blocking path that leads to a stable state.*

Theorem 2 extends RV's theorem to the incomplete-information environment. The intuition and the formal proof are delegated to Subsections 4.3-4.4. So far the Theorem states only the existence of a desired learning-blocking path. In the rest of this subsection, we discuss some properties of such a matching process[21]

Firstly, the construction of the desired learning-blocking path is summarized in Figure 1, where the output is a stable state. Both the allocation and the information structure evolve along the path. On the one hand, when the allocation changes, agents' payoff may either increase or decrease, so are the salaries between two matched agents. With more construction details in Subsections 4.3-4.4, we will see a series of *ad hoc* monotonic changes in agents' payoffs.

On the other hand, no matter whether the status quo is blocked or not, the partition profile gets (weakly) finer and finer along the process, i.e., agents get more and more

[19] As we mentioned in Footnote 10, salaries must be rounded to the nearest dollar, penny, or mill. This is a technical assumption to ensure *finite* bargaining choices when a worker-firm pair negotiates, as well as, more importantly, a realistic situation in decentralized market practice which we aim to describe. See Crawford and Knoer (1981), Kelso and Crawford (1982), and Chen et al. (2016) for similar integral assumptions when *finite* matching processes are studied. In marriage models where our results hold and there is no payment involved, of course this assumption is not necessary any more.

Moreover, one can easily construct an example, in which (i) two firms compete for one worker, (ii) the salary increment converges to zero, and (iii) the limit salary still permits a blocking. Therefore, without Assumption 1, finite path cannot be guaranteed even in the complete-information environment.

[20] Stable states are well defined under Assumption 1 and the existence is still guaranteed (see (Crawford and Knoer, 1981, Theorem 1)). In the rest of the section, we refer blocking, stability and $\mathscr{S}(\mathbf{w}^*)$ as the ones defined under Assumption 1.

[21] See Subsection 5.3 for discussions on the limit of learning-blocking paths, i.e., the set of stable states that can be achieved by those paths.

information. This is due to agents' perfect recall of the history. The additional information agents can learn comes from both historical event and historical direct observation. See Subsection 5.2 for more discussions about perfect recall.

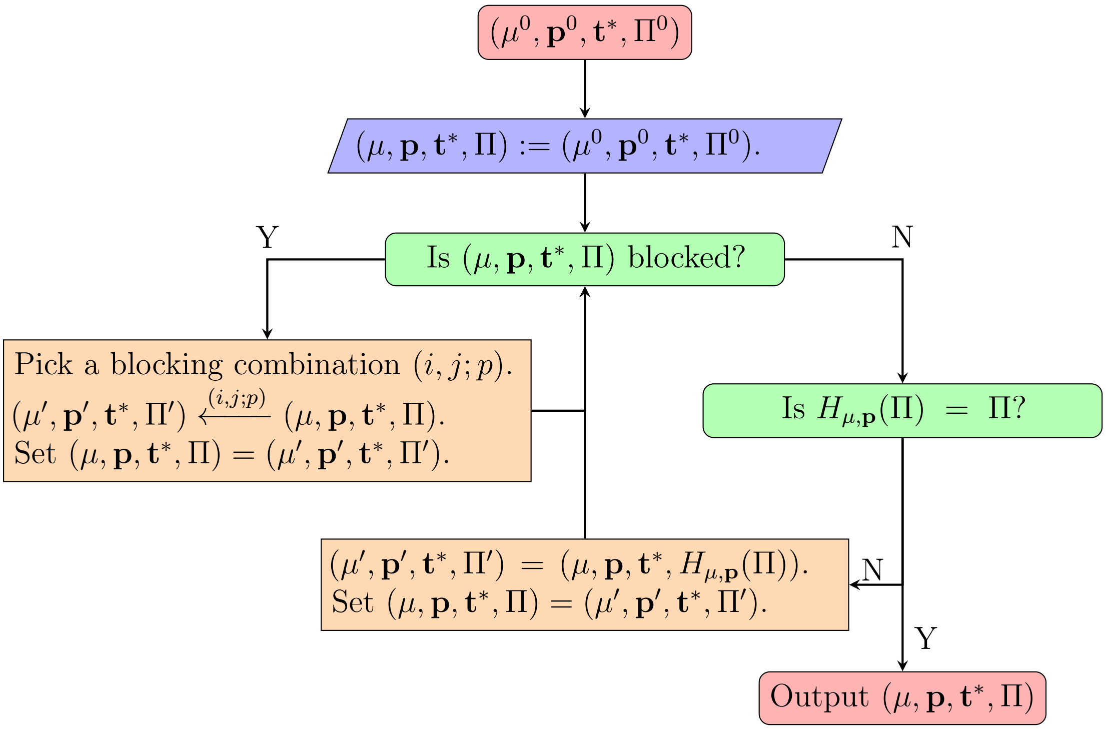

Figure 1: An illustration of the desired learning-blocking path.

Secondly, every stable state could obviously be achieved by the learning-blocking path starting at *itself*.[22] A natural question is whether we can achieve every stable state by learning-blocking paths starting at *a common* initial state.

In the complete-information setting, RV observes that every stable matching can be achieved by a blocking path starting with a no-match status (every individual is unmatched). Intuitively, given any stable matching, replicating the stable matching pair by pair constitutes a desired blocking path. With incomplete information, however, the observation analogous to RV's is not true. To see this, consider a market with one firm and one worker where a match at some proper salary constitutes a stable state. However, the firm concerns the worker's counterfactual worst type and may not be willing to be matched with him. In this case, a no-match initial status cannot lead to the stable states with a match.

4.2.2 Random Path to Stability

Now we consider a random process which starts with an arbitrary state. The process proceeds to generate a random learning-blocking path, i.e., whenever an intermediate

[22] As a corollary, every stable allocation in $\mathscr{S}(\mathbf{w}^*)$ (defined in (15)) could be achieved by the learning-blocking path starting from *itself*.

state is blocked by many combinations, the process randomly satisfies one of them. In particular, the blocking combination to be satisfied is drawn from a distribution whose support include all blocking combinations. The distribution assigns each combination a strictly positive probability and the probability depends only on the state but not on the rematching history (i.e., independent distributions for different intermediate states). This random process mimics the matching behavior in the real world labor market: agents meet and negotiate randomly until they expect no more (utility or profit) improvement.

The way in which we randomly draw blocking combinations is almost the same as what RV did, except for the difference in informational environments. With incomplete information, we extend RV's random convergence result below, which follows immediately from Theorem 2.

Theorem 2′. *Suppose that Assumption 1 holds. Then the random learning-blocking path starting from an arbitrary state converges with probability one to a stable state.*

Although we can extend RV's random convergence result, one observation from RV is not true here. In particular, RV observes that a random blocking path starting at a no-match initial status will achieve *every* stable matching with strictly positive probability. With incomplete information, however, this observation is not true for the same reason as why we cannot achieve every stable state by deterministic learning-blocking paths starting a common initial state.[23]

4.3 A Comparison of Theorem 2 and RV's Theorem

Theorem 2 is parallel to RV's theorem. RV constructs a sequence of subsets of agents, $\{A(l)\}_{l=1}^{r-1}$ and correspondingly a sequence of matchings $\{\mu^l\}_{l=1}^{r}$ such that at each step l, there is no blocking pair for μ^{l+1} that is contained in $A(l)$. Thus, a blocking pair for μ^{l+1}, if any, must involve at least one agent outside $A(l)$. In the subsequent step $l+1$, the set $A(l+1)$ is obtained by the union of $A(l)$ and one of those outside agents. As a result, the sequence $A(l)$ expands. Since there are only finitely many agents, the sequence $\{A(l)\}_{l=1}^{r-1}$ will reach the set which includes everyone in the market, i.e., $A(r-1) = I \cup J$. The construction of such a sequence implies that there is no blocking pair for μ^r that is contained in $A(r-1)$, i.e., μ^r is stable.

[23]In the complete-information setting, Ma (1996) considers a random order mechanism that puts agents into an empty room one by one according to a queue of individual agents. The queue is randomly drawn from all possible orders of individual agents with equal probability. Once an agent in the front of the queue enters the room, the room is closed and a random matching process takes place within the room. The room closes until the matching in the room is stable with respect to the set of players in the room. Then the next agent enters the room and etc.

The random order mechanism of Ma (1996) differs from RV's randomized matching mechanism only in one aspect. RV's random queue, which starts at a no-match status, allows for pairs in the queue, while Ma's random queue allows only individual agents. This difference between two mechanisms leads to significant different implications. RV's mechanism implies that every stable matching can be achieved with a positive probability, while Ma (1996) shows by example that *not* every stable matching can be obtained with a positive probability.

Particularly, when RV constructs μ^{l+2}, they need to guarantee that there is no blocking pair for μ^{l+2} that is contained in $A(l+1)$. To do this, they choose for the outside agent her/his favorite partner in $A(l)$ among those who are willing to form a blocking pair with the outsider; for the agent dumped by the chosen one (if any), they choose her/his favorite partner in $A(l)$ among those who are willing to form a blocking pair with the dumped agent; and etc. Such a dumping chain will exhaust the blocking opportunities within $A(l+1)$ and will produce a desired μ^{l+2}.

When firms' information is incomplete, it is no longer clear which worker is their favorite one. In particular, firms do not know which employee to favor among those who are willing to form a blocking pair, as they may not know these workers' types. More precisely, a firm may not be willing to form a blocking pair with its *de facto* favorite worker in the worry of his worst possible type, which the firm considers possible.

The property that no blocking pair for μ^{l+1} is contained in $A(l)$ plays a key role in the proof of RV's theorem. With incomplete information, firms in $A(l)$ observe either that there is a rematching or that there is no rematching, both of which may lead to information updating. This implies that there may be blocking pairs for μ^{l+1} that is contained in $A(l)$ when μ^{l+2} is derived, because of information updating. As a result, RV's argument does not apply here: information updating invalidates RV's construction.

In view of these two issues, our proof proceeds with two key ideas, which are presented in Lemma 1 and Lemma 2 respectively. Lemma 1 addresses the first issue by sidestepping the task of directly picking the favorite partner. Instead, we find the favorite partner step by step, where in each step the underlying agent only needs to find a better partner.

More importantly, Lemma 2 shows the existence of a learning-blocking path that conducts the following tasks: (i) to include more agents so that no blocking pair arises within the enlarged set; and (ii) to ensure that at least one firm is matched with a worker whom it has never matched with, whenever the first task fails. To see why such a construction would address the second issue and help prove Theorem 2 (see Subsection 4.4 for the formal proof), we first note that Case (ii) can happen only finitely many times. We sketch the proof of Theorem 2 before we present Lemma 1 and Lemma 2.

Sketch for the Proof of Theorem 2. The proof of Theorem 2, which is by construction. Like RV's algorithm, our algorithm also produces a sequence of subsets of agents which contains no internal blocking pairs. Whenever Case (ii) of Lemma 2 happens, we output an emptyset, which certainly contains no internal blocking pairs. Thus, the subsets to be produced may either expand or shrink to an empty set along the sequence. We have argued that Case (ii) can happen only finitely many times. Therefore, the shrinkage due to Case (ii) can happen only finitely many times. As will be clear in the formal proof, the subsets may also shrink because of the observation of no rematching, which again implies that at least one firm is matched with a worker whom it has never matched with. Our algorithm allows only these two kinds of set shrinkage. Since we have finite agents, this will be enough to ensure that the sequence of subsets containing no internal blocking pair will ultimately expand to include all agents. □

Remark 2. *Informally, the idea we use to prove Theorem 2 is "trying" to replicate the construction algorithm of RV. The key point is to identify information updatings such that ultimately there is no information updating and thus the (modified) construction algorithm of RV could work through. The modified RV algorithm differs from the original algorithm only in the following sense: in the original algorithm, agents go for a better partner by comparing the realized payoffs from different matches; while in the modified one, agents go for a better partner by evaluating the worst possible payoffs from different matches.*

4.3.1 Best Partner

First, instead of finding the "best" partner, we introduce a "lost mate finding rule" , which decomposes the problem of finding the best partner into small steps. The "lost mate finding rule" is applicable to both complete- and incomplete-information environments. To be precise, consider the following example, where all irrelavent ingredients are omitted.

Example 4. *Consider a matching with three pairs, i.e., $\mu^0(\alpha) = a$, $\mu^0(\beta) = b$ and $\mu^0(\gamma) = c$, where every agent has a positive payoff. All other ingredients of the market are omitted for simplicity. Suppose (α, b) is a blocking pair, satisfying which leads to a matching with $\mu^1(\alpha) = b$, $\mu^1(\beta) = \emptyset$ and $\mu^1(\gamma) = c$. Suppose now (α, c) is a blocking pair. Then satisfying (α, c) leads to another matching with $\mu^2(\alpha) = c$, $\mu^2(\beta) = \emptyset$ and $\mu^2(\gamma) = \emptyset$. It must be true that (β, b) is a blocking pair for μ^2 as they are both unmatched (with payoff zero), but previously matched with positive payoffs. In other words, firm b can find his previously lost mate, i.e., worker β, which leads to a matching with $\mu^3(\alpha) = c$, $\mu^3(\beta) = b$ and $\mu^3(\gamma) = \emptyset$.*

Since (α, c) is a blocking pair for μ^1, we know that (α, c) must be a blocking pair for μ^0 because the payoff for α under μ^1 is higher than that under μ^0. Moreover, firm c is α's favorite partner among those who are willing to form a blocking pair with him under μ^0. Therefore, deriving μ^3 via the blocking chain above with the "lost mate finding rule" is as if deriving μ^3 by choosing for worker α a favorite agent directly.

Lemma 1 below formalizes this idea of finding the best blocking partner by small steps. Moreover, for one side of the market, agents' payoffs are unchanged except that one agent's payoff strictly increases.

Lemma 1. *Fix a state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, a subset of agents $A \subset I \cup J$ whose partners under μ^0 are all in A, and a worker $i^0 \notin A$ (resp. a firm $j^0 \notin A$). Suppose that every matched agent in A has a strictly positive payoff, and that $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ is blocked by i^0 (resp. j^0) and a firm (worker) in A. Then there exists a finite learning-blocking path $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^{L}$ such that the following three statements are true.*

1. *For each firm $j \in A$ (resp. worker $i \in A$), the worker-firm pair (i^0, j) (resp. (i, j^0)) is not a blocking pair for the state $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$.*
2. *Worker $(\mu^0)^{-1}(\mu^L(i^0)) \in A$ (resp. firm $\mu^0((\mu^L)^{-1}(j^0))$) becomes unmatched under μ^L and firm $\mu^L(i^0) \in A$ (resp. worker $(\mu^L)^{-1}(j^0)$) gets a strictly higher payoff. Moreover, for all $i, j \in A$ such that $i \neq (\mu^0)^{-1}(\mu^L(i^0))$ and $j \neq \mu^L(i^0)$ (resp. for all $i, j \in A$ such that $i \neq (\mu^L)^{-1}(j^0)$ and $j \neq \mu^0((\mu^L)^{-1}(j^0))$), their match and thus payoff under $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ is the same as under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$.*
3. *For agents in $A \cup \{i^0\}$ (resp. $A \cup \{j^0\}$), their partners under μ^L are all in $A \cup \{i^0\}$ (resp. $A \cup \{j^0\}$).*

In Example 4, $\{\beta, \gamma, b, c\}$ corresponds to the set A and worker α corresponds to the outside worker i^0. The worker who becomes unmatched is γ and the firm who gets a strictly higher payoff is firm c. Moreover, for β and b, their match and thus payoff under μ^3 is the same as under μ^1.

Since the two cases in Lemma 1 are symmetric, we focus on the case with i^0. The learning-blocking path to prove Lemma 1 is constructed by Algorithm 1 below. Along the algorithm, k counts the number of rematchings; $\bar{i}$, $\bar{j}$ and $\bar{p}$ are other state variables, where $(\bar{i}, \bar{j}; \bar{p})$ is the match to be restored by the "lost mate finding rule," as the blocking pair (β, b) for μ^2 in Example 4 (with some proper salary).

Algorithm 1

Input. A state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, a subset A of $I \cup J$ and a worker i^0.

Initialization. Initialize $\bar{i}$, $\bar{j}$, and $\bar{p}$ to be the dummy agent $\emptyset$, an arbitrary firm and an arbitrary real number respectively. Initialize k to be 0.

Process. Consider two mutually exclusive cases.

(a) If there exists no blocking combination $(i, j; p)$ for $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ such that $i = i^0$ and $j \in A$, then go to End.

(b) Otherwise, arbitrarily pick a blocking combination $(i, j; p)$ for the state $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ such that $i = i^0$ and $j \in A$. Derive $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$ such that $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1}) \xleftarrow{(i,j;p)} (\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$. Consider two mutually exclusive cases.

   i. If $\bar{i} \notin I$, derive k' such that $k' = k + 1$.

   ii. If $\bar{i} \in I$, derive $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2})$ such that $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2}) \xleftarrow{(\bar{i},\bar{j};\bar{p})} (\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$, i.e., let $\bar{j}$ "find her lost mate" at the previous salary. Derive k' such that $k' := k + 2$.

   Set $(\bar{i}, \bar{j}, \bar{p})$ to be $((\mu^k)^{-1}(j), j, \mathbf{p}^k_{i,\bar{j}})$, and k to be k'. Go to Process.

End. Output $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^{L}$, where $L = k$.

Remark 3. *Lemma 1 enables an alternative proof for RV's theorem if there is no incomplete information and transfer, also an alternative proof of (Chen et al., 2010, Theorem 1) if information is complete.* Algorithm 1 *imposes little restriction when we choose blocking combinations once Case (b) is triggered, so that the proof does not rely on the "favorite mate picking rule" used in RV's proof, or the "initiator getting the lion's share of the resulting surplus" rule introduced by Chen et al. (2010) (also used in the algorithm of Chen et al. (2016)).*

Proof of Lemma 1. We prove the statements with input i^0; the case with j^0 is symmetric. The proof is by three steps.

Step 1. Algorithm 1 *produces a learning-blocking path.*

It suffices to argue that whenever Case (b)-ii is triggered, $(\bar{i}, \bar{j}; \bar{p})$ is a blocking combination for the underlying state $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$; then for each $l = 1, \ldots, L$, $(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$ is derived from $(\mu^{l-1}, \mathbf{p}^{l-1}, \mathbf{w}^*, \Pi^{l-1})$ by satisfying a blocking combination for $(\mu^{l-1}, \mathbf{p}^{l-1}, \mathbf{w}^*, \Pi^{l-1})$, which means $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^L$ is a learning-blocking path.

To see the argument, we note that any $(\bar{i}, \bar{j}, \bar{p})$ in Case (b)-ii must be defined as $(\bar{i}, \bar{j}, \bar{p}) = ((\mu^k)^{-1}(j), j, \mathbf{p}^k_{\bar{i},\bar{j}})$ for some k. Then by the definition of $\xleftarrow{(i,j;p)}$, we know $\mu^{k+1}(\bar{i}) = \emptyset$ and $\mu^{k+1}(i^0) = \mu^{k+1}(i) = j = \bar{j}$. Note that $\bar{i}$ was initialized to be $\emptyset$. Thus if $k = 0$, Case (b)-i is triggered after we derive $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$. This implies that when Process-(b) is triggered next time, $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2})$ will derived. Therefore, the combination $(\bar{i}, \bar{j}, \bar{p})$ will be satisfied in the Case (b)-ii where we derive $(\mu^{k+3}, \mathbf{p}^{k+3}, \mathbf{w}^*, \Pi^{k+3})$. By the condition of deriving $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2})$, i.e., i^0 is a blocking worker, we know that $(\mu^{k+2})^{-1}(\bar{j}) = \emptyset$. Note that $\mu^{k+2}(\bar{i}) = \mu^{k+1}(\bar{i}) = \emptyset$. Therefore, both $\bar{i}$ and $\bar{j}$ are unmatched under μ^{k+2}. Moreover, since $k = 0$, $\bar{p} = \mathbf{p}^0_{\bar{i},\bar{j}}$. By the assumption of the lemma, the matched payoffs for $\bar{i}$ and $\bar{j}$ under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ are both strictly positive. Hence, $(\bar{i}, \bar{j}, \bar{p})$ is a blocking combination for $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2})$.

If $k \geq 1$, Case (b)-ii is triggered after we derive $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$. This implies that when Process-(b) is triggered next time, $(\mu^{k+3}, \mathbf{p}^{k+3}, \mathbf{w}^*, \Pi^{k+3})$ will derived. Therefore, the combination $(\bar{i}, \bar{j}, \bar{p})$ will be satisfied in the Case (b)-ii where we derive $(\mu^{k+4}, \mathbf{p}^{k+4}, \mathbf{w}^*, \Pi^{k+4})$. The rest of the argument is similar to the case with $k = 0$, except that we need to show $\bar{p} = \mathbf{p}^0_{\bar{i},\bar{j}}$. This is obvious by inductively noticing that for any matched pair under $(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$ which is contained in A, the salary within the pair is the same as when the pair was mathed under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$.

Step 2. $L < \infty$, *i.e., the learning-blocking path constructed by* Algorithm 1 *is finite.*

To see this, note that whenever we trigger Process, worker i^0's payoff strictly increases. Since we assumed the finiteness of match surplus and integer payments, worker i^0's payoff can increase with only finitely many rematchings, i.e., Process is triggered finitely many times. Since at most two new states are derived whenever Process is triggered, we know that the learning-blocking path is finite.

Step 3. The three statements in the lemma are true.

The first one is trivial by the stopping condition of ALGORITHM 1, i.e., the condition in Case (a). The third one is also trivial because only blocking pairs contained in $A \cup \{i^0\}$ are satisfied throughout ALGORITHM 1. For the second statement, we note that the terminal state $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ must be either $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ or be a state derived in Case (b)-ii. Obviously, $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ satisfies the second statement. The following is true by induction in $k > 1$: whenver a state $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2})$ is derived in Case (b)-ii (the lost mate of some unmatched firm has been found), only one firm and one worker in A have a different match from $(\mu^0, \mathbf{p}^0)$, which are firm $\mu^{k+2}(i^0)$ and worker $(\mu^k)^{-1}(\mu^{k+2}(i^0)) = (\mu^0)^{-1}(\mu^{k+2}(i^0))$ respectively. Then worker $(\mu^0)^{-1}(\mu^{k+2}(i^0))$ becomes unmatched under μ^{k+2}. The payoff of firm $\mu^{k+2}(i^0) = \mu^{k+1}(i^0)$ strictly increased when we were deriving $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$. This completes the proof. □

4.3.2 IDENTIFYING INFORMATION UPDATING

The second idea, unlike the first one, is specific to the incomplete-information environment. Formally, we say that a set of agents A *contains no internal blocking pair under state* $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ if no pair or individual in A blocks $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$. Consider a set of agents that contains no internal blocking pair under a state. We can construct a finite learning-blocking path that either (i) leads to a larger set with no internal blocking at the updated state or (ii) ensures that that at least one firm is matched with a worker whom it has never matched with. The former case is in line with the proof of RV, whereas the latter case will trigger a particular strict information updating (strictly finer partition profile), which takes the form we illustrate below.

EXAMPLE 5. *(Example 3 revisited.) Under the initial state $(\mu, \mathbf{p}, \mathbf{w}^2, \Pi)$, firm b is not sure about worker α's type, which is expressed by*

$$|\{\mathbf{w}(\alpha) : \mathbf{w} \in \Pi_b(\mathbf{w}^2)\}| = |\{\mathbf{w}^2, \mathbf{w}^3\}| \geq 2.$$

However, under the state $(\mu', \mathbf{p}', \mathbf{w}^2, \Pi')$, firm b is matched with worker α and thus knows the type of α, which is expressed as

$$|\{\mathbf{w}(\alpha) : \mathbf{w} \in \Pi'_b(\mathbf{w}^2)\}| = |\{\mathbf{w}^2\}| = 1.$$

In words, firm b updates its information strictly after it is matched with a worker whom it has never met before and, more importantly, whose type was not pinned down before.

The following lemma formalizes and ensures such a strict information updating, whenver task (i) fails. The presumed situation requires that A contains no internal blocking pair.

LEMMA 2. *Fix a state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, a subset of agents $A \subset I \cup J$ whose partners under μ^0 are all in A, and a worker $i^0 \notin A$ (resp. a firm $j^0 \notin A$). Suppose that A contains* *no internal blocking pair for $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, that every matched agent in A has a strictly positive payoff, and that $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ is blocked by i^0 (resp. j^0) and a firm (worker) in A. Then there exists a finite learning-blocking path $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^L$ such that for agents in $A \cup \{i^0\}$ (resp. $A \cup \{j^0\}$), their partners under μ^L are all in $A \cup \{i^0\}$ (resp. $A \cup \{j^0\}$), and that one of the following statements is true.*

1. *$A \cup \{i^0\}$ (resp. $A \cup \{j^0\}$) contains no internal blocking pair under $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$.*
2. *Under $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$, there exists a firm $\hat{j}$ who learns the type of a worker which it did not know at the initial state, i.e.,*

$$|\{\mathbf{w}(({\mu^L})^{-1}(\hat{j})) : \mathbf{w} \in \Pi^0_{\hat{j}}(\mathbf{w}^*)\}| \geq 2 \textit{ but } |\{\mathbf{w}(({\mu^L})^{-1}(\hat{j})) : \mathbf{w} \in \Pi^L_{\hat{j}}(\mathbf{w}^*)\}| = 1.$$

We prove Lemma 2 by construction. To construct the learning-blocking path, we (apply Lemma 1 to) match i^0 with his best partner in A among those who are willing to form a blocking pair with i^0, which is denoted by j^1; if $(\mu^0)^{-1}(j^1) \neq \emptyset$, then we (apply Lemma 1 to) match $(\mu^0)^{-1}(j^1)$ to his best partner in A among those who are willing to form a blocking pair with the worker $(\mu^0)^{-1}(j^1)$, which is denoted by j^2; and so on. Then, at some step t, we have either that j^t was unmatched under μ^0, i.e., $(\mu^0)^{-1}(j^t) = \emptyset$; or that j^t was matched to a worker $(\mu^0)^{-1}(j^t)$ under μ^0 but $(\mu^0)^{-1}(j^t)$ has no blocking opportunity with firms in A. Now we check whether or not there is a blocking pair in $A \cup \{i^0\}$ for the underlying state. Clearly, if the answer is yes, then we are in Case (i). The main argument in Lemma 2 shows that if there exists a blocking pair in $A \cup \{i^0\}$ for the underlying state, then we are in Case (ii).

The learning-blocking path for the proof of Lemma 2 is constructed by ALGORITHM 2 below. We focus on the case with $i^0 \notin A$ since the algorithm for the other case is symmetric. Along the algorithm, k counts the number of rematchings and m counts the number of PROCESS being triggered. Apart from k and m, we also keep track of the state variable α which corresponds to i^0, $(\mu^0)^{-1}(j^1)$, $(\mu^0)^{-1}(j^2)$,..., illustrated above and will be updated during the matching process.

ALGORITHM 2

INPUT. A state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, a subset A of $I \cup J$ and a worker $i^0 \notin A$.

INITIALIZATION. Initialize α to be i^0 as the outside worker. Set k and m both to be 0.

PROCESS. Derive m' such that $m' = m + 1$. Set m to be m'. Run ALGORITHM 1, with input $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ and α. Denote the output by $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=k}^{K+k}$, where $K+1$ is the length of the learning-blocking path produced by ALGORITHM 1. Denote $(\mu^0)^{-1}(\mu^{K+k}(\alpha))$ by i^m. Derive k' such that $k' = K + k$, and set k to be k'. Consider three mutually exclusive cases.

(a) If $A \cup \{i^0\}$ contains no blocking pair for $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$, derive A' such that $A' = A \cup \{i^0\}$. Go to END.

(b) If there exists a blocking combination $(i, j; p)$ for $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ such that $i = i^m$ and $j \in A$, then go to PROCESS with $\alpha = i^m$.

(c) Otherwise, arbitrarily pick a blocking combination $(i, j; p)$ such that $\{i, j\} \subset A \cup \{i^0\}$. Derive $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$ such that $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1}) \xleftarrow{(i,j;p)} (\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$. Set A' to be $\varnothing$. Derive k' such that $k' = k + 1$, and set k to be k'. Go to END.

END. Output A' and $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^{L}$, where $L = k$.

Proof of Lemma 2. We first prove Lemma 2 for the case with $i^0 \notin A^0$. This is done by four steps. Then we complete the argument for the case with $j^0 \notin A^0$ as the fifth step.

Step 1. ALGORITHM 2 *always produces a learning-blocking path.*

Note that new states are derived either by ALGORITHM 1, which produces a learning-blocking path, or in Case (c), where we satisfy a blocking combination for the status quo. Therefore, ALGORITHM 2 always produces a learning-blocking path.

Step 2. For agents in $A \cup \{i^0\}$, their partners under μ^L are all in $A \cup \{i^0\}$.

By hypothesis of the lemma, for agents in A, their partners under μ^0 are all in A. Note that new states are derived either by ALGORITHM 1 or in Case (c), both of which satisfy only blocking pairs contained in $A \cup \{i^0\}$. Hence, under μ^L, agents in $A \cup \{i^0\}$ are either unmatched or matched with someone in $A \cup \{i^0\}$.

Step 3. The learning-blocking path produced by ALGORITHM 2 *is finite.*

Along the path, new states are derived either by Case (c) or by ALGORITHM 1. Since at most one new state is derived by the former (after which the algorithm ends immediately), we only need to focus on the finiteness of states generated by the latter. By Lemma 1, each learning-blocking path produced by ALGORITHM 1 is finite. Therefore, it suffices to show that ALGORITHM 1 can be triggered at most finitely many times.

By Lemma 1, each time we run ALGORITHM 1, all firms' payoffs are unchanged except for the firm $\mu^{K+k}(\alpha)$ who gets strictly higher payoff. Since the surplus from each match is a real number (thus bounded) and I is finite, we know that for each firm the possible payoffs are bounded. By Assumption 1, only integer payments are permitted. Therefore, for each firm strictly higher payoff can be obtained only finitely many times. Note further that J is finite. Hence, the firm-side payoff-increasing that is due to ALGORITHM 1 can happen at most finitely many times, i.e., ALGORITHM 1 can be triggered at most finitely many times, which implies that the path produced by ALGORITHM 2 is finite.

Step 4. One of the two arguments in the lemma is true.

On the one hand, if the path ends by Case (a), i.e., $A' = A \cup \{i^0\}$, then A' contains no internal blocking pair for $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ by construction. On the other hand, suppose the path ends by Case (c), where we derive the final state $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ from $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$ by satisfying $(i, j; p)$. Obviously, $\mu^L(i) = j$. Then firm j observes the type of worker i, i.e., for every $\mathbf{w} \in \Pi_j^L(\mathbf{w}^*)$ we have $\mathbf{w}(i) = \mathbf{w}^*(i)$, which implies $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi_j^L(\mathbf{w}^*)\}| = 1$. Then it suffices to show that $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi_j^0(\mathbf{w}^*)\}| \geq 2$.

Suppose to the contrast that $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi_j^0(\mathbf{w}^*)\}| = 1$, i.e., for every $\mathbf{w} \in \Pi_j^0(\mathbf{w}^*)$, it is true that $\mathbf{w}(i) = \mathbf{w}^*(i)$. Then j has complete information about i's type at state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. Obviously, $i \neq i^m$; otherwise Case (b) is triggered instead of Case (c). Moreover, $i \neq i^{m-1}$; otherwise the underlying $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=k}^{K+k}$ was not an output of the Algorithm 1 with the input worker $i = i^{m-1}$, since $(i, j; p)$ is a blocking combination for $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k}) = (\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$. We distinguish two cases.

Case 4.1. $i \notin \{i^0, i^1, \ldots, i^{m-2}\}$.

We proceed to argue that $(i, j; p)$ was a blocking combination for the state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. Since $i \notin \{i^0, i^1, \ldots, i^{m-2}\}$, worker i was neither an input of Algorithm 1, nor $(\mu^0)^{-1}(\mu^{K+k}(\alpha))$ for some input worker $\alpha \in \{i^0, i^1, \ldots, i^{m-2}\}$, where $K + k$ indexes the output of the Algorithm 1 with the input worker α. By Lemma 1, we know that worker i's match and thus payoff keeps unchanged along the path $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^{L-1}$. Particularly, worker i's payoff under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ is the same as his payoff under $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$. Since all firms in A get weakly higher payoffs whenever we run Algorithm 1, firm j's payoff was worse at $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ than at $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$. Since $(i, j; p)$ is a blocking combination for the state $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$ and j has complete information about i's type at $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, we know that $(i, j; p)$ must be a blocking combination for $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. However, $\{i, j\} \subset A \cup \{i^0\}$ and $i \neq i^0$ imply that $\{i, j\} \subset A$, a contradiction.

Case 4.2. $i \in \{i^0, i^1, \ldots, i^{m-2}\}$.

Without loss of generality, suppose $i^{m-2} = i$. After we run Algorithm 1 with input i, there was no further blocking pair that is contained in A and involves i (Lemma 1). Particularly, $(i, j; p)$ was not a blocking combination for the state $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$, where $K + k$ indexes the output of the Algorithm 1 with the input worker i. Since $i = i^{m-2}$, we have $K + k < L - 1$ due to another round of Algorithm 1 with the input worker i^{m-1}.

We proceed to argue that $(i, j; p)$ was a blocking combination for $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$, which will be a contradiction. By our hypothesis, $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi_j^0(\mathbf{w}^*)\}| = 1$. Since the partition profile gets finer along any learning-blocking path, $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi_j^{K+k}(\mathbf{w}^*)\}| = 1$, i.e., j has complete information about i's type at state $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$. Since $i \neq i^{m-1}$ and $i \neq i^m$, we know by Lemma 1 that worker i's match and thus payoff keeps unchanged along the path $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=K+k}^{L-1}$, which is the output of Algorithm 1 with

i^{m-1} as the input worker. Particularly, worker i's payoff under $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$ is the same as his payoff under $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$. Since all firms in A get weakly higher payoffs whenever we run ALGORITHM 1, firm j's payoff was worse at state $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$ than at state $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$. Since $(i, j; p)$ is a blocking combination for the state $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$ and j has complete information about i's type at state $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$, we know that $(i, j; p)$ must be a blocking combination for $(\mu^{K+k}, \mathbf{p}^{K+k}, \mathbf{w}^*, \Pi^{K+k})$. This completes the argument.

Step 5. The lemma holds for the case with $j^0 \notin A^0$.

The firm-version ALGORITHM 2 to construct a desired learning-blocking path for this case is defined analogously by interchanging workers and firms. The constructed learning-blocking path is finite for symmetric reasons as in the case with $i^0 \notin A^0$. The first statement of the lemma is straightforward when the path ends with Case (a). To show the statement when the path ends with Case (c), we consider the blocking combination $(i, j; p)$ for $(\mu^{L-1}, \mathbf{p}^{L-1}, \mathbf{w}^*, \Pi^{L-1})$. Suppose that j had complete information about i's type at $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. When $j = j^0$, we can argue that $(i, j; p)$ was a blocking combination for $(\mu^K, \mathbf{p}^K, \mathbf{w}^*, \Pi^K)$, where K indexes the output of the firm-version ALGORITHM 1 with the input firm j. This contradicts $(\mu^K, \mathbf{p}^K, \mathbf{w}^*, \Pi^K)$ being an output of the firm-version ALGORITHM 1. When $j \neq j^0$, we can argue that $(i, j; p)$ was a blocking combination for $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, which is a contradiction to $\{i, j\} \subset A$ since A contains no internal blocking pair. □

4.4 PROOF OF THEOREM 2

In this subsection, we prove Theorem 2, which formalizes the sketch we provided at the beginning of Subsection 4.3. Without loss of generality, we assume that the initial state is individually rational. Otherwise, finitely many breaking-ups will lead to an individually rational state. The learning-blocking path is constructed by the following algorithm. Along the algorithm, k counts the number of market states; and more importantly, in any stage of the algorithm, the set A contains no internal blocking pair.

ALGORITHM 3

INPUT. An arbitrary individually rational state: $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$.

INITIALIZATION. Set k to be 0. Set A to be $\varnothing$.

PHASE 1. There are two mutually exclusive cases.

(a) $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ is blocked. Go to PHASE 2.

(b) $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ is not blocked. Go to PHASE 3.

PHASE 2. There are two mutually exclusive cases.

(a) If $i \notin A$ and $j \notin A$ for any blocking combination $(i, j; p)$ for $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$, then arbitrarily pick a blocking combination $(i, j; p)$. Derive $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$ such that $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1}) \xleftarrow{(i,j;p)} (\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$.

i. If $A \cup \{i, j\}$ contains no internal blocking pair for the state $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$, then derive A' such that $A' = A \cup \{i, j\}$ and set A to be A'. Derive k' such that $k' = k + 1$ and set k to be k'. Go to PHASE 1.

ii. If $A \cup \{i, j\}$ contains an internal blocking pair (i', j') for the state $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$, where the blocking salary is p', then derive $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2})$ such that $(\mu^{k+2}, \mathbf{p}^{k+2}, \mathbf{w}^*, \Pi^{k+2}) \xleftarrow{(i',j';p')} (\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$. Derive k' such that $k' = k + 2$ and set k to be k'. Set A to be $\varnothing$. Go to PHASE 1.

(b) Otherwise, run ALGORITHM 2 with input $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ and i (resp. j), where $(i, j; p)$ is a blocking combination for $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ such that $i \notin A$ and $j \in A$ (resp. $i \in A$ and $j \notin A$). Denote the output by A' and $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=k}^{K+k}$, where K is the length of the learning-blocking path produced by ALGORITHM 2. Derive k' such that $k' = K + k$ and set k to be k'. Set A to be A'. Go to PHASE 1.

PHASE 3. There are two mutually exclusive cases.

(a) If $H_{\mu^k, \mathbf{p}^k}(\Pi^k) = \Pi^k$, go to END.

(b) If $H_{\mu^k, \mathbf{p}^k}(\Pi^k) \neq \Pi^k$, derive Π^{k+1} such that $\Pi^{k+1} = H_{\mu, \mathbf{p}}(\Pi^k)$. Derive $(\mu^{k+1}, \mathbf{p}^{k+1})$ such that $(\mu^{k+1}, \mathbf{p}^{k+1}) = (\mu^k, \mathbf{p}^k)$. Derive k' such that $k' = k + 1$ and set k to be k'.

i. If $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ is not blocked, then go to PHASE 3.

ii. If $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ is blocked by some combination $(i, j; p)$, then derive $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$ such that $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1}) \xleftarrow{(i,j;p)} (\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$. Derive k' such that $k' = k + 1$ and set k to be k'. Set A to be $\varnothing$. Go to PHASE 1.

END. Output $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^{L}$, where $L = k$.

Proof of Theorem 2. The proof is by three steps.

Step 1. ALGORITHM 3 *always produces a learning-blocking path.*

Note that new states are derived either in Case 2-(b), or at the beginning of Case 3-(b) or in Case 3-(b)-ii. Since both of the new states in Case 3-(b) (if any) are derived according to the rules which we used to define learning-blocking paths, it suffices to show that the sequence $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=k}^{K+k}$ in Case 2-(b) is a learning-blocking path. Since the sequence is produced by ALGORITHM 2, it then suffices to verify the conditions of Lemma 2 whenever we trigger Case 2-(b). In other words, we will show by induction that each agent in A is either unmatched or matched with another agent in A, that the set A contains no internal blocking pair, that every matched agent in A has a strictly positive payoff, and that the status quo is blocked by i and a firm in A.

Firstly, A is either updated to be empty or nonempty. When A is updated to be empty, it is trivially true that each agent in A is either unmatched or matched with another agent in A. On the other hand, we consider each Case 2-(a)-i and those Case 2-(b) such that A is updated to be nonempty. Consider Case 2-(a)-i. By induction hypothesis, each agent in A is either unmatched or matched with another agent in A. Note that $\mu^{k+1}(i) = j$. Thus, each agent in $A \cup \{i, j\}$ is either unmatched or matched with another agent in $A \cup \{i, j\}$. For Case 2-(b), we note that ALGORITHM 2 outputs either $A \cup \{i\}$ (resp. $A \cup \{j\}$) or $\varnothing$. By Lemma 2, each agent in $A \cup \{i\}$ (resp. $A \cup \{j\}$) is either unmatched or matched with another agent in $A \cup \{i\}$ (resp. $A \cup \{j\}$).

Secondly, we argue A contains no internal blocking pair. This is trivial when A is updated to be empty. For each Case 2-(a)-i, the triggering condition is that $A \cup \{i, j\}$ contains no internal blocking pair. Again, we are done. For those Case 2-(b) such that ALGORITHM 2 outputs $A \cup \{i\}$ (resp. $A \cup \{j\}$), we know that $A \cup \{i\}$ (resp. $A \cup \{j\}$) contains no internal blocking pair, by Lemma 2 and the induction hypothesis that conditions of Lemma 2 hold.

Thirdly, we argue that every matched agent in A has a strictly positive payoff, for which we only need to consider possibilities where A is updated to be nonempty. In Case 2-(a)-i, since (i, j) is a blocking pair, their matched payoffs are both strictly positive. By induction hypothesis, every matched agent in A has a strictly positive payoff. Therefore, every matched agent in $A \cup \{i, j\}$ has a strictly positive payoff. For those Case 2-(b) such that ALGORITHM 2 outputs $A \cup \{i\}$ (resp. $A \cup \{j\}$), we know, by Lemma 2 and the same induction hypothesis as above, that every matched agent in $A \cup \{i\}$ (resp. $A \cup \{j\}$) has a strictly positive payoff.

Lastly, it is trivial that the status quo in Case 2-(b) is blocked by i and a firm j in A, which is the triggering condition.

Step 2. $L < \infty$, i.e., the learning-blocking path produced by ALGORITHM 3 *is finite.*

We first claim that the set A is updated to be empty set at most finitely many times. Note that the set A is updated to be empty only in one of the following three cases: Case 2-(a)-ii, Case 2-(b), or Case 3-(b)-ii. By Lemma 2, whenever A is updated to be empty in Case 2-(b), a firm is matched with a worker who it has never met, which can happen at most $|I| \times |J|$ many times. For Case 3-(b)-ii, we know that firm j has never been matched with worker i before $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$; otherwise, $(\mu^{k-1}, \mathbf{p}^{k-1}, \mathbf{w}^*, \Pi^{k-1})$ is blocked, contradicting the triggering condition of Phase 3. Therefore, Case 3-(b)-ii can be triggered at most $|I| \times |J|$ many times. Finally for Case 2-(a)-ii, we know that firm j' has never been matched with worker i' before $(\mu^{k+1}, \mathbf{p}^{k+1}, \mathbf{w}^*, \Pi^{k+1})$; otherwise, $(\mu^k, \mathbf{p}^k, \mathbf{w}^*, \Pi^k)$ is blocked by (i', j', p'), but either $i' \in A$ or $j' \in A$ contradicts the triggering condition of Case 2-(a). Therefore, Case 2-(a)-ii can be triggered at most $|I| \times |J|$ many times. To sum up, the set A is updated to be empty set at most finitely many times.

To see the finiteness of L, note that the set A can either shrink (to empty set) or expand. We have shown that it shrinks only finitely many times. Obviously, the expanding direction is finite because of the finiteness of $I \cup J$ and the finiteness of the shrinking direction. Therefore, both the expanding direction and the shrinking direction are finite, which implies that the L is finite.

Step 3. $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ *is a stable state.*

Since the Algorithm 3 ends only if Phase 3 is triggered, which implies that $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ is not blocked. The ending condition in Case 3-(a) implies $H_{\mu^L, \mathbf{p}^L}(\Pi^L) = \Pi^L$. Therefore, $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ is stable. □

5 Discussions

5.1 The Information Structure of Stable States

In this subsection, we investigate the information structure of stable states, particularly the structure of the partition profiles that support a common allocation as a stable allocation. Intuitively, the following monotonicity property holds: less information (coarser partition profile) makes it easier for a state to be stable.

Given a stable state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, if we blur the information, i.e., replace Π with a coarser partition profile Π', the resulted state $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ is individually rational and, by Fact 1, not blocked. However, $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ may not be stable any more because of information instability, i.e., the third requirement of Definition 3 is not satisfied. Nevertheless, the $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ is "essentially" stable. Formally, we say that a state $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ is *quasi-stable* if $(\mu, \mathbf{p}, \mathbf{w}, H^k_{\mu,\mathbf{p}}(\Pi'))$ is not blocked for all k. Since the iteration of $H^k_{\mu,\mathbf{p}}(\cdot)$ stops at a fixed point in finitely many steps, quasi-stability of $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ implies that $(\mu, \mathbf{p}, \mathbf{w}, H^k_{\mu,\mathbf{p}}(\Pi'))$ is stable (Definition 3) for some k.

Fact 2. *Suppose* $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ *and* $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *are two states such that* Π' *is coarser than* Π*. If* $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is stable, then* $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ *is quasi-stable.*[24]

Proof of Fact 2. Obviously, the individual rationality of $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is equivalent to the individual rationality of $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$. Since $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable and thus not blocked, we

[24]Fact 2 captures the intuition that if a matching is stable under some information structure then it is also stable for every coarser information structure. In a setting different from our paper, (Chakraborty et al., 2010, Theorem 1) shows that if a matching *mechanism* is stable under some information structure it is also stable for every coarser information structure.

know by Fact 1 that $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ is not blocked. Similarly, $(\mu, \mathbf{p}, \mathbf{w}', \Pi')$ is not blocked for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$. (Otherwise, $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ is blocked, contradicting $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$.) Therefore, $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}) \subset N^{(\mu,\mathbf{p},\Pi')}(\mathbf{w})$. As a result, $[H_{\mu,\mathbf{p}}(\Pi')]_k(\mathbf{w}') = \Pi'_k(\mathbf{w}')$ for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$ and all $k \in I \cup J$.

Since the state $(\mu, \mathbf{p}, \mathbf{w}', \Pi')$ is not blocked for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$, we know that $(\mu, \mathbf{p}, \mathbf{w}', H_{\mu,\mathbf{p}}(\Pi'))$ is not blocked for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$, which implies that $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}) \subset N^{(\mu,\mathbf{p},H_{\mu,\mathbf{p}}(\Pi'))}(\mathbf{w})$. Keep applying this argument and define $\Pi'^k := H_{\mu,\mathbf{p}}(\Pi'^{k-1})$ for $k = 1, 2, \dots,$ until we find a fixed point of $H_{\mu,\mathbf{p}}$. This must be done within finitely many times because the partition profile gets finer whenever it is not a fixed point. Denote the fixed point by Π'^∞. Then induction shows that $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}) \subset N^{(\mu,\mathbf{p},\Pi'^\infty)}(\mathbf{w})$ and that $(\mu, \mathbf{p}, \mathbf{w}', \Pi'^\infty)$ is not blocked for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$, particularly for $\mathbf{w}' = \mathbf{w}$. □

5.2 Robustness of Convergence

The learning-blocking path described in subsection 4.1 includes three kinds of inference that can be drawn from different observations. However, the firms may not be as sophisticated as we modeled. For example, they may not be able to keep track of the partition profile Π which represents the information structure of the entire market. It is also possible that firms do not have perfect recall as we implicitly assumed in subsection 4.1. Alternatively, they may be more sophisticated in coming up with arguments to rule out more type profiles than we would like them to.

In general, we can think of an *information updating pattern* as three mappings $H_{\mu,\mathbf{p}}$, $B_{\mu,\mathbf{p},\Pi;i,j,p}$ and $P_{\mu,\mathbf{p},\Pi;i,j,p}$ that specify agents' information updating for each type of observations: $H_{\mu,\mathbf{p}} : \Pi \longmapsto \Pi'$ is defined by (5) for firms under the case of no re-matching, $P_{\mu,\mathbf{p},\Pi;i,j,p} : \Pi_j \longmapsto \Pi'_j$ is for j under the case of rematching of $(i, j; p)$, and $B_{\mu,\mathbf{p},\Pi;i,j,p} : \Pi_{j'} \longmapsto \Pi'_{j'}$ is for firm $j' \neq j$ under the case of rematching by $(i, j; p)$.

When firms are not as sophisticated as we have modeled, $P_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_j)$ and $B_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_{j'})$ are coarser than those updated partitions defined by (16)-(17). Theorem 2 holds if we use a coarser information updating pattern along the learning-blocking path, as long as $P_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_j)$ and $B_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_{j'})$ are finer than Π'_j and $\Pi'_{j'}$, respectively, where

$$\begin{aligned} \Pi'_j(\mathbf{w}) =& \Pi_j(\mathbf{w}) \cap O^{\{i\}}(\mathbf{w}) && \text{for all } \mathbf{w} \in \Omega; \\ \Pi'_{j'}(\mathbf{w}) =& \Pi_{j'}(\mathbf{w}) && \text{for all } \mathbf{w} \in \Omega \text{ and all } j' \neq j. \end{aligned}$$

Actually we allow for any temporary change to the partition profile, which may be coarser or finer, as long as the change happens only finitely many times.

Firms may also be more sophisticated than we have modeled. For example, when myopic agents can observe not only the allocation change but also firms' offer making and workers' response, they may experience some intermediate stage between two allocations. During those intermediate stages, firms may have the access to more information. In this case, one can enrich an information updating pattern by considering more possible observations. Reformulating the convergence problem under the alternative information updating patterns may be an interesting extension of our analysis.

5.3 Limits of Convergent Learn-Blocking Paths

In Theorem 2, there may be many convergent learning-blocking paths, which lead to different stable states. Intuitively, those limit states depend on the initial state. For example, when the initial state is stable, the limit is uniquely pinned down. This subsection is to show that the dependence is in a more sophisticated way.

Consider an initial state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ that is individually rational. We define

$$\mathscr{S}(\mathbf{w}^*, \Pi^0) := \left\{ (\mu, \mathbf{p}) \in \mathscr{A} : \begin{array}{c} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi^0) \text{ is not blocked} \\ \text{and } \forall k, (\mu, \mathbf{p}, \mathbf{w}^*, H^k_{\mu,\mathbf{p}}(\Pi^0)) \text{ is not blocked} \end{array} \right\},$$

where $H^k_{\mu,\mathbf{p}}(\cdot)$ is the k-iteration of the operator $H_{\mu,\mathbf{p}}(\cdot)$. By definition, if $(\mu, \mathbf{p}) \in \mathscr{S}(\mathbf{w}^*, \Pi^0)$, then $(\mu, \mathbf{p}, \mathbf{w}^*, H^\infty_{\mu,\mathbf{p}}(\Pi^0))$ is a stable state. Then $\mathscr{S}(\mathbf{w}^*, \Pi^0)$ is interpreted as the set of allocations that can be essentially supported by $(\mathbf{w}^*, \Pi^0)$ as stable allocations.

Obviously, $\mathscr{S}(\mathbf{w}^*, \Pi^0) \subset \mathscr{S}(\mathbf{w}^*)$, where $\mathscr{S}(\mathbf{w}^*)$ is defined in (15). Theorem 2 guarantees that starting at $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, we can find a finite learning-blocking path such that the resulted stable allocation lies in the set $\mathscr{S}(\mathbf{w}^*)$. Moreover, we claim that for any such learning-blocking path, the resulted stable allocation must be in $\mathscr{S}(\mathbf{w}^*, \Pi^0)$, which depends on the initial state, particularly the initial information structure.

Fact 3. *For any finite learning-blocking path that starts at $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and leads to a stable state $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$, we have $(\mu^L, \mathbf{p}^L) \in \mathscr{S}(\mathbf{w}^*, \Pi^0)$.*[25]

Proof. Since $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^L)$ is stable, it is not blocked. According to the definition of a learning-blocking path, the partition profile Π^L is finer than Π^0. Hence, Fact 1 implies that $(\mu^L, \mathbf{p}^L, \mathbf{w}^*, \Pi^0)$ is not blocked.

Similarly, for every $\mathbf{w} \in N^{(\mu^L, \mathbf{p}^L, \Pi^L)}$, we know that $(\mu^L, \mathbf{p}^L, \mathbf{w}, \Pi^0)$ is not blocked. That is, $N^{(\mu^L, \mathbf{p}^L, \Pi^0)} \supset N^{(\mu^L, \mathbf{p}^L, \Pi^L)}$. Therefore, when we restrict attention to the set $N^{(\mu^L, \mathbf{p}^L, \Pi^L)}(\mathbf{w}^*)$, Π^L_j is finer than $[H_{\mu,\mathbf{p}}(\Pi^0)]_j$ for all j. As a result, for every $\mathbf{w} \in N^{(\mu^L, \mathbf{p}^L, \Pi^L)}(\mathbf{w}^*)$, we know that $(\mu^L, \mathbf{p}^L, \mathbf{w}, H_{\mu,\mathbf{p}}(\Pi^0))$ is not blocked. By induction on k, we know that every $k = 1, 2, \ldots$, $(\mu^L, \mathbf{p}^L, \mathbf{w}, H^k_{\mu,\mathbf{p}}(\Pi^0))$ is not blocked for all $\mathbf{w} \in N^{(\mu^L, \mathbf{p}^L, \Pi^L)}(\mathbf{w}^*)$, particularly for $\mathbf{w}^*$. This completes the proof. $\square$

We close this subsection by considering under what condition the matching process we constructed converges to complete-information stable allocation. One possible condition is imposed on the initial state, particularly on the initial information structure. To be

[25] Since the statement of the fact already assumes the existence of a desired learning-blocking path, we do not need Assumption 1 here.

precise, we consider a particular starting point $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. If $\mathscr{S}(\mathbf{w}^*, \Pi^0)$ contains only complete-information stable allocations, then the output allocation of our learning-blocking path will be complete-information stable. There may be other conditions imposed on, say, the set of all possible type-assignments Ω, which we leave as an open question.

5.4 Bayesian Stability

One crucial assumption we made is that firms evaluate blocking combinations according to the worst-case payoff that is compatible with their information. We inherited this assumption from LMPS to (i) make our analysis clean; and more importantly to (ii) have a clear comparison between two stability notions (see Theorem 1). However, this assumption is neither necessary for defining stable states nor for showing the convergence of learning-blocking paths.

In this subsection, we first introduce a prior distribution and propose a Bayesian analogy of Definitions 1-3. It will be clear later that we can either have common prior or heterogeneous prior. Then we compare our Bayesian stability with the stability notion studied in Liu (2017).[26] Finally, we discuss the convergence of learning-blocking paths.

5.4.1 The Definition of Bayesian Stability

Suppose it is commonly known that the type-assignment is drawn from a distribution G, where the support of G is Ω. For a subset Ω' of Ω, we use the notation $G(\cdot|\Omega')$ to denote the conditional distribution of type assignment functions restricted on Ω'. The notation $\tilde{\mathbf{w}}$ is for a random variable drawn from G or $G(\cdot|\Omega')$. Now with this common prior, a market state is described by $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ and G. We fix G throughout this subsection.

The Bayesian stability notion that we propose has three requirements: (i) the state is individually rational, (ii) the state is not Bayesian blocked, and (iii) the fact of individual rationality and no Bayesian blocking pair provides no further information to agents. Individual rationality of a state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is defined as in Definition 1.

To define a blocking combination $(i, j; p)$ for $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, we first note that agents evaluate each other by the conditional expectation of the rematching payoff, conditional on each agent's own information. Particularly, if i and j consider a switch to each other from their partners under μ, the conditional expected premuneration value for worker i is simply $\nu_{\mathbf{w}(i),\mathbf{f}(j)}$. For firm j, the willingness of worker i to participate the deviation helps it to refine its consideration set. Formally, define

$$D_j^{(i,j;p)} := \left\{ \tilde{\mathbf{w}} \in \Pi_j(\mathbf{w}) : \nu_{\tilde{\mathbf{w}}(i),\mathbf{f}(j)} + p > \nu_{\tilde{\mathbf{w}}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \right\},$$

[26] Bayesian stability is also studied in Bikhchandani (2017), where the set of Bayesian stable outcomes is defined by iteratively elimination of Bayesian blocked outcomes. Unlike the notions of Liu (2017) and our paper, the existence of a stable outcome is not necessarily guaranteed (see Proposition 3 and Example 2 of Bikhchandani (2017)).

where the dependence of $D_j^{(i,j;p)}$ on the public information $(\mu, \mathbf{p}, \Pi)$ is suppressed in the notation. Then firm j only needs to consider type-assignments in $D_j^{(i,j;p)}$ and its the conditional expected premuneration value is $\mathbb{E}\left[\phi_{\tilde{\mathbf{w}}(i),\mathbf{f}(j)}|D_j^{(i,j;p)}\right]$.

Definition 5. *A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be **Bayesian blocked** if there exists a worker-firm pair (i,j) and a payment $p \in \mathbb{R}$ such that i is not matched with j under μ, and they prefer each other at p to their current match, i.e.,*

$$\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ and}$$
$$\mathbb{E}\left[\phi_{\tilde{\mathbf{w}}(i),\mathbf{f}(j)}|D_j^{(i,j;p)}\right] - p > \phi_{\mathbf{w}(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j}.$$

Similar to subsection 3.3, we use a binary partition of Ω to distinguish two events: a state with $(\mu, \mathbf{p}, \Pi)$ is individually rational and not Bayesian blocked; or otherwise. The binary partition is formally denoted by $N^{(\mu,\mathbf{p},\Pi;G)}$. For two type-assignments $\mathbf{w}'$ and $\mathbf{w}''$, $N^{(\mu,\mathbf{p},\Pi;G)}(\mathbf{w}') = N^{(\mu,\mathbf{p},\Pi;G)}(\mathbf{w}'')$ if and only if one of the following conditions hold:

(i) $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ is either Bayesian blocked or not individually rational, and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ is either Bayesian blocked or not individually rational;

(ii) $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ is individually rational and not Bayesian blocked, and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ is individually rational and not Bayesian blocked.

Then the information contained in "the fact of individual rationality and no blocking pair" (and/or its compliment "the fact of not being individual rational or being blocked") is summarized in $N^{(\mu,\mathbf{p},\Pi;G)}$. Finally, aggregating this piece of information leads to a new partition profile $H_{\mu,\mathbf{p},G}(\Pi)$ defined as follows.

$$[H_{\mu,\mathbf{p},G}(\Pi)]_j(\mathbf{w}') := \Pi_j(\mathbf{w}') \cap N^{(\mu,\mathbf{p},\Pi;G)}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{ and all } j \in J.$$

Definition 6. *Fix a prior G. A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be **Bayesian stable** if*

1. *it is individually rational,*
2. *it is not Bayesian blocked, and*
3. *Π is a fixed point of $H_{\mu,\mathbf{p},G}$, i.e. $H_{\mu,\mathbf{p},G}(\Pi) = \Pi$.*

The existence of a Bayesian stable state is obviously guaranteed by the existence of a complete-information stable state.

5.4.2 A Comparison of Definition 6 and Liu's Notion

Recently, Liu (2017) studied stable matching and stable beliefs in a Bayesian setting, which also introduces a Bayesian notion of stability. See Definition 4 and Definition 10 in Liu (2017). Overall, the two stability notions have different emphases but they complements each other.

The main difference between two models is the set of observables. Firms in our setting can observe the type of their own employee; while firms in Liu (2017) cannot. Moreover, firms in our setting have partial information about other firms' heterogeneous information, which is reflected in the publicly known partition profile; while firms in Liu (2017) have no idea about other firms' heterogeneous information. The difference in observables ensures that in Liu (2017), firms cannot make inference from the public signal of not being blocked. Naturally, the stability notions of Liu (2017) require individual rationality and no blocking pair, but not information stability as we did in our Definition 3.

Another more important difference between two stability notions is the object investigated, which can be seen as a result of the difference in observables. To be precise, we fix a prior distribution. The object in Liu (2017) is an allocation. Liu's notion emphasizes the consistency between the prior distribution and the allocation. However, the object in our Bayesian setting is a combination of an allocation and a partition profile. Our notion emphasizes the consistency between the partition profile and the allocation, where the partition profile is part of the solution as is the allocation itself.

However, when we consider the trivial partition profile, i.e., $\Pi_j = \{\emptyset, \Omega\}$ for all j, then our Definiteion 6 is the same as Liu's Definition 4 when either of the following two conditions holds:

(i) $\Omega = W^I$; or

(ii) firms cannot observe the type of their own employee.

Therefore, the two stability notions are for different scenarios but coincide when we eliminate the setup difference.

5.4.3 Convergence of Learning-Blocking Paths

Similar to Theorem 2-2′, we can show the convergence of learning-blocking paths in the Bayesian environment. To be precise, the algorithm to construct the particular finite learning-blocking path differs from Algorithm 1-3 only in that we need to replace blocking combinations by Bayesian blocking combinations and replace the information operator $H_{\mu,\mathbf{p}}$ by $H_{\mu,\mathbf{p},G}$. The formal statements and proof details are almost repetitions of those for Theorem 22-2′, and thus omitted.

6 Concluding Remarks

In this paper, we introduce a notion of stability for matching with one-sided incomplete information which accommodates firm-specific information. Moreover, we show the convergence of random learning-blocking paths to stable states; the convergence extends the result due to Roth and Vande Vate (1990) to an incomplete-information environment. In proving the result, we provide a new proof for RV's theorem. By applying the "lost mate finding rule," the proof avoids the intermediate optimization problems. It makes the proof more elementary, as we illustrated in subsection 4.3. Furthermore, it is crucial for the current paper to describe how firms form and update their possibilistic information, and how they utilize the information to draw inferences. From this perspective, our analysis provides a benchmark for studying the strategic foundations of stability in a decentralized matching market such as Lauermann and Nöldeke (2014).[27]

In our paper, as well as in LMPS, there is only one-sided incomplete information. It is certainly important to study matching markets with two-sided incomplete information, which involves a subtle formulation of the agents' higher-order reasoning.[28]

Our stability notion and the convergence result in this paper also provide a theoretical benchmark for further study of matching markets with dynamics. For example, in Chen and Hu (2017) we investigate the monotonicity properties of the set of stable matchings as the population increases.[29]

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[27]LMPS leaves the study of the strategic foundations of stability as an open question. The difficulty of the study can be seen from the following two remarks from LMPS.

First, "...agents *make inferences* from intermediate outcomes during the matching process, so the set of possible incomplete-information stable outcomes becomes a 'moving target.' Providing decentralized foundations for both complete- and incomplete-information stable matchings is an open and obviously interesting problem" (LMPS, p. 543).

And second, "...Under incomplete information, the connection between stable matches and the process by which stable matches are formed is yet less obvious. In the process of encountering others and accepting or rejecting matches, the agents are likely to *learn* about their environment. As a result, the information structure prevailing at the end of the matching process will typically differ from that at the beginning..." (LMPS, p. 570).

[28]See Chen and Hu (2017) for details.

[29]We show by examples that even in matching markets with one-sided incomplete information, the monotonicity properties fail. However, these properties can be restored when we restrict the original allocation (before the population increase) to complete-information stable allocations. See, for example, Blum et al. (1997) for an analogous study in the complete-information setting.

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