LEARNING BY MATCHING*

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Abstract

This paper studies stability and matching processes in the job market with incomplete information on worker side. Each agent is associated with a type, which determines their payoffs from a match. Moreover, firms' information structure is described by partitions over possible worker type profiles. With this flexible and firm specific information, we propose a stability notion capturing, in addition to individual rationality and no blocking pair, the idea that absence of rematching conveys no further information. When an allocation is not stable under the status quo information structure, a new pair of allocation and information structure will be derived. We show that starting from an arbitrary allocation and an arbitrary information structure, the process of allowing randomly chosen blocking pairs to rematch, accompanied with information updating, will converge to an allocation that is stable under the updated information structure with probability one. Our results are robust with respect to various alternative learning patterns.

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1 Introduction

Matching is one of the important functions of markets (Roth (2008)). This paper focuses on stability and matching processes in the job market setting.[1] We depart from the prevailing assumption in two-sided matching theory that information is complete, i.e., the characteristic of all market participants are common knowledge. In particular, we study incomplete information on the worker side. We first describe how firms form and update their possibilistic information about workers' types, and propose a incomplete-information stability notion that allows for flexible information structure. Then we show that a random matching process converges to an allocation that is stable under the updated information structure, with probability one.

Stability under complete information requires individual rationality, i.e., each agent has nonnegative payoff, and no blocking pair, i.e., no worker-firm pair would prefer matching with each other at a certain wage to staying with the current matching.[2] In contrast, in a job market with incomplete information on the worker side, a typical firm may not know the types of their potential employees which determine their productivity. Without the information, however, the firm does not know if she would prefer another worker to her current partner. As a result, the notion of blocking pair and stability in the complete-information environment is no longer appropriate.

Following Liu et al. (2014) (LMPS for short), we assume that firms who are uncertain about their potential employees' types care about the worst possibility, and that a firm can observe the type of her own employee. Unlike LMPS, however, we describe the firms' heterogeneous information by a profile of partition over the set of type profiles of the workers. Given an information structure, we propose a stability notion which extends the notion of stable matching with incomplete information proposed by LMPS.[3] In our setting, a state of the market consists an allocation (i.e., a matching together with a prevailing wage profile) and an information structure. A state is stable if the allocation is individually rational and admits no blocking pair with respect to the information structure; and moreover, the absence of rematching conveys no further information to the firms. The last requirement, in particular, formalizes "informational stability" which is specific to the incomplete information setting.

[1]Stable matchings have been connected to both equity and efficiency in resource allocation, two of most important objectives in economics. See Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) for how stability implies elimination of justified envy, a basic fairness axiom. See Shapley and Shubik (1971) and Liu et al. (2014) for how stability implies efficiency.

[2]We take the job market setting (with transferable utility) to facilitate the comparison between our stability notion and that of Liu et al. (2014). Nevertheless, our convergence result (Theorems 1-1′), can be established without difficulty in models with non-transferable utility, such as the ex ante stability notion of Bikhchandani (2017).

[3]Specifically, our notion of stability coincides with the notion proposed LMPS when the only source of firms' heterogeneous information is that each firm can observe the type of her own employee. (See Subsection 5.2 for details).

Equipped with the notion of stability, we then study a matching process which mimics the individual behavior of searching for desired job, school and life partner. Indeed, if a worker and a firm find themselves better off matching with each other than maintaining the status quo, they will match together to make an improvement. The new matching may again admit a blocking pair and thus a rematching opportunity, which results in another new matching, and so on. One prominent question is whether such a process finally stops at a stable matching.[4]

When information is incomplete, along the matching process each observation of rematching or no rematching provides information to the firms. Consequently, even if no rematching is observed under an information structure, there may be some rematching opportunity with updated information. Therefore, a matching process is necessarily associated with an information updating process where firms draw inference from each observation.

In this paper, we consider a specific information updating process. Specifically, we classify the firms' observations into following three groups: that a rematching is not observed, that a rematching of some pair is observed, and that within each pair the firm observes the type of the worker. Information updating along with each observation refines each firm's information partition.

For an arbitrary initial market state, this learning and rematching process consists of a sequence of states. We call it a LEARNING-BLOCKING PATH. Our main result shows that by suitably choosing the blocking pairs, a LEARNING-BLOCKING PATH reaches after a finite number of rematchings an allocation which is stable with respect to the updated information. The finite number depends on the number of agents but not the number of worker type profiles. This construction implies that when blocking pairs are randomly selected to rematch, the resulting LEARNING-BLOCKING PATH converges to a stable state with probability one. Our result also holds under various alternative learning patterns.[5]

The rest of this section reviews related literature. Section 2 introduces the model. Section 3 defines stability with incomplete information. Section 4 describes a specific learning pattern along a LEARNING-BLOCKING PATH and presents our convergence results. Robustness of convergence, the connection between our stability notion and that of LMPS, and efficiency of stable states are discussed in Section 5. Section 6 concludes.

[4]Knuth (1976) provides an example of a blocking path that admits a cycle, i.e. any matching on the path is not stable. This motivates the study of the convergence of blocking paths. We provide a similar example in the job market model, i.e., a one-to-one matching model with transfers, in Appendix A. Pursuing an affirmative answer to the above question results in a fruitful literature. See Roth and Vande Vate (1990), Kojima and Ünver (2008), Klaus and Klijn (2007b), Chen et al. (2016) and Fujishige and Yang (2016).

[5]For example, agents may ignore/forget the information conveyed in some observations or draw more sophisticated inference from the observations. See Subsection 5.1 for more discussion.

Related Literature

The seminal model of Gale and Shapley (1962), studying the marriage market (college admission market), has been used in a large literature of two-sided matching. Many classical theories are surveyed in Roth and Sotomayor (1990) and more recently by, for example, Roth (2008). In this literature, a prevailing assumption is that the information is complete.

Recently, LMPS introduces a notion of incomplete-information stability. Our stability notion coincides with theirs when the only source of firms' heterogeneous information is that each firm can observe the type of her own employee. To be precise, take an incomplete-information stable matching outcome in their setting, where an outcome consists of an allocation and a worker type profile. Stability of LMPS can be interpreted as a result of iteration that starts with a state consisting of the stable matching outcome and a *particular* partition profile. The particular partition profile distinguishes for each firm only the type of her own employee (if any), i.e., each firm puts all worker type profiles with the same type for her employee (if any) into the same cell; the iteration refines the initial partition profile using the information conveyed by the fact that the state is not blocked. The iteration starting with the *particular* partition profile finally stops at a stable state in our setting.[6] In contrast to LMPS, we allow for *flexible* partition profiles in stable states. Our flexible information structure for firms facilitates the study of the matching process.

The stability notion of Bikhchandani (2017) is similar to that of LMPS but focuses on non-transferable utility and Bayesian stability, instead of the worst case desideratum. Pomatto (2015) considers a matching game and derives the same incomplete-information stable outcomes as LMPS by using forward induction reasoning. Chakraborty et al. (2010) studies college admission markets where the students' characteristics are unobservable for colleges but colleges may receive signals about these characteristics. They propose and identify a stable matching *mechanism* which specifies a matching and how much information to reveal to the agents, given each reported signal profile with an information structure. They also make clear that stability of a matching mechanism should depend on the agents' information structure.[7] In a similar vein, we define a stable allocation with respect to different information structures of the agents.

Whether a matching process converges to a stable allocation is known as the problem of finding paths to stability. The first result on paths to stability is shown in marriage markets with complete information by Roth and Vande Vate (1990) (hereafter, RV). Chen et al. (2010) investigate path to stability in models with complete information and

[6]See Remark 6 for details.

[7]To the best of our knowledge, Chakraborty et al. (2010) is the first paper that internalizes available information into stability notions. Another stream of literature studies incomplete information about other's preferences. See, for example, Roth (1989) and Ehlers and Massó (2007).

transferable utilities.[8] All these results involve no private types or information updating, which is crucial in the current paper. Our result implies an alternative proof for RV's theorem, as well as Theorem 1 of Chen et al. (2010). Bikhchandani (2017) discusses path to stability under a Bayesian notion of stability. In his paper, the final matching outcome of a blocking path is Bayesian stable *conditional on history*. Namely, along a blocking path, agents cannot block with any of their erstwhile partners. We do not impose such a restriction here. Lazarova and Dimitrov (2017) also studies paths to stability with incomplete information, under a permissive blocking notion that enables agents to learn the types of other agents on the opposite side of the market as long as there is possibility of learning by satisfying permissive blocking pairs. However, their approach is not applicable when more conservative blocking notions are adopted, such as the notions in LMPS, Pomatto (2015) and Bikhchandani (2017).

2 The Model

We follow LMPS in considering the following setup of matching with incomplete information. The setup generalizes the complete-information matching models studied by Shapley and Shubik (1971) and Crawford and Knoer (1981).

There is a finite set I of workers to be matched with a finite set J of firms. Denote a generic worker by i and a generic firm by j. Each agent's index is publicly observed. The productivity of an agent are however determined by the agent's *type*. Let $W \subset \mathbb{R}$ be the finite set of worker types and $F \subset \mathbb{R}$ be the finite set of possible firm types. The type assignment function for firms is denoted by $\mathbf{f} : J \to F$, while the type assignment function for workers is denoted by $\mathbf{w} : I \to W$. We denote by Ω a set of worker type assignment functions, i.e., $\Omega \subset W^I$.

A match between worker type $w \in W$ and firm type $f \in F$ gives rise to the *worker premuneration value* $\nu_{wf} \in \mathbb{R}$ and *firm premuneration value* $\phi_{wf} \in \mathbb{R}$.[9] The sum of the premuneration values $\nu_{wf} + \phi_{wf}$ is called the *surplus of the match*. Denote these values by $\nu_{\mathbf{w}(\emptyset),\mathbf{f}(j)}$ for the unmatched worker and $\phi_{\mathbf{w}(i),\mathbf{f}(\emptyset)}$ for the unmatched firm, both of which are set to be zero. The functions $\nu : W \times F \to \mathbb{R}$ and $\phi : W \times F \to \mathbb{R}$ are common knowledge.

Given a match between worker i (of type $\mathbf{w}(i)$) and firm j (of type $\mathbf{f}(j)$), the worker's payoff and the firm's payoff are, respectively, $\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p$ and $\phi_{\mathbf{w}(i),\mathbf{f}(j)} - p$, where $p \in \mathbb{R}$ is the payment made to worker i by firm j.[10]

[8] The main result of Chen et al. (2010), partially incorporated into Chen et al. (2016), is the convergence of blocking paths to *competitive equilibrium*, which is stronger than stability. See also Fujishige and Yang (2016).

[9] See Mailath et al. (2013, 2017) for discussions on premuneration values.

[10] If we take the intuition that salaries must be rounded to the nearest dollar, penny, or mill, the analysis in this section will go through without any extra difficulty. This more practical restriction will

A *matching* is a function $\mu : I \to J \cup \{\emptyset\}$, one-to-one on $\mu^{-1}(J)$, that assigns worker i to firm $\mu(i)$, where $\mu(i) = \emptyset$ means that worker i is unemployed and $\mu^{-1}(j) = \emptyset$ means that firm j does not hire a worker.

A *payment scheme* $\mathbf{p}$ associated with a matching μ is a vector that specifies a payment $\mathbf{p}_{i,\mu(i)} \in \mathbb{R}$ for each $i \in I$ and $\mathbf{p}_{\mu^{-1}(j),j} \in \mathbb{R}$ for each $j \in J$. To avoid nuisance cases, we associate zero payments with unmatched agents, setting $\mathbf{p}_{\emptyset j} = \mathbf{p}_{i\emptyset} = 0$.

An *allocation* $(\mu, \mathbf{p})$ consists of a matching μ and an associated payment scheme $\mathbf{p}$. We assume that the entire allocation is publicly observable and that within a matched pair the firm can observe the worker's type. Denote by $\mathscr{A}$ the set of allocations.

As LMPS, we assume that $\mathbf{f}$ is commonly known, namely, each firm's type is common knowledge.[11] Each worker knows his own type, but other agents only know that the worker type assignment function $\mathbf{w}$ belongs to Ω.

The firms may have imprecise or wrong information about the realized type assignment function. A typical firm j's information is described by a partition Π_j of Ω. For any $\mathbf{w} \in \Omega$, write $\Pi_j(\mathbf{w})$ as the element of the partition that contains $\mathbf{w}$. Each $\mathbf{w}' \in \Pi_j(\mathbf{w}) \in \Pi_j$ is a possible type assignment from firm j's viewpoint when the realized type assignment function is $\mathbf{w}$. Denote the profile of partitions by Π, i.e. $\Pi := (\Pi_1, \ldots, \Pi_{|J|})$.

A *state* of the matching market, $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, specifies an allocation $(\mu, \mathbf{p})$, a type assignment function $\mathbf{w}$ and a partition profile Π such that each firm's partition is consistent with her partner's type observed by her, i.e., for every j and every $\mathbf{w}'$, $\mathbf{w}''(\mu^{-1}(j)) = \mathbf{w}'(\mu^{-1}(j))$ for all $\mathbf{w}'' \in \Pi_j(\mathbf{w}')$.

3 Stability with Incomplete Information

3.1 Individual Rationality

A state is said to be individually rational if each agent receives at least the payoff of remaining unmatched which is zero.

Definition 1. *A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be **individually rational** if*

$$\nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \geq 0 \text{ for all } i \in I \text{ and}$$
$$\phi_{\mathbf{w}(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \geq 0 \text{ for all } j \in J.$$

be imposed in Section 4, where we study a matching process.

[11]It is possible to extending our stability notion to the two-sided incomplete-information environment, where $\mathbf{f}$ is not common knowledge just as $\mathbf{w}$. See Chen and Hu (2017) for details.

3.2 Blocking

The notion of incomplete-information "blocking" naturally extends its complete-information counterpart. In particular, a matching is blocked if some worker-firm pair (i, j) can mutually benefit from matching with each other. Following LMPS, we assume that a firm is concerned about the worst case of a worker, as she does not know his true type.

Definition 2. *A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be **blocked** if there exists a worker-firm pair (i, j) and a payment $p \in \mathbb{R}$ such that*

$$\begin{aligned} \nu_{\mathbf{w}(i),\mathbf{f}(j)} + p &> \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ and} && (1) \\ \phi_{\mathbf{w}'(i),\mathbf{f}(j)} - p &> \phi_{\mathbf{w}'(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} && (2) \end{aligned}$$

for all $\mathbf{w}' \in \Pi_j(\mathbf{w})$ satisfying

$$\nu_{\mathbf{w}'(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}. \tag{3}$$

For firm j to participate a potential blocking pair for the state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, she must guarantee an improvement for every relevant type assignment function. More precisely, when firm j considers forming a potential *blocking pair* (i, j) with worker i, at some potential salary p, a type assignment function $\mathbf{w}'$ is relevant for firm j when $\mathbf{w}' \in \Pi_j(\mathbf{w})$ and (3) holds. All type assignment functions violating (3) are irrelevant because the rematching agreement can never be reached due to the worker's objection.

The following fact is an immediate consequence of the worst-case desideratum in Definition 2. It says that the existence of a blocking pair is easier with more precise information.

Fact 1. *Suppose that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ and $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ are two states such that Π' is a finer partition profile than Π. If $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is blocked, then $(\mu, \mathbf{p}, \mathbf{w}, \Pi')$ is blocked.*

3.3 Stability

When $\Pi_j(\mathbf{w}') = \{\mathbf{w}'\}$ for all $j \in J$ and $\mathbf{w}' \in \Omega$, i.e., information is complete, a stable state is simply a state that is individually rational and not blocked. The notion of incomplete-information "stability" differs from its complete-information counterpart in the following way: When information is complete, stable matching conveys the intuition that when "the agents have a very good idea of one another's preferences and have easy access to each other,... we might expect that stable matchings will be especially likely to occur" (Roth and Sotomayor, 1990, pp. 22). In contrast, the partition Π_j in our setting describes only firm j's imprecise idea about the workers' information. The "stability" notion that we propose below captures: i) the state is individually rational,

ii) the state is not blocked, and, most importantly, iii) it is common knowledge that the state is individually rational and not blocked. The first two properties coincide with the complete-information stability, while the third one describes "information stability" which is specific to the incomplete-information environment.

Given the public information of a state $(\mu, \mathbf{p}, \Pi)$, let $N^{(\mu,\mathbf{p},\Pi)}$ be the partition of Ω such that $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}') = N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}'')$ if and only if either neither $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ nor $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ is blocked or both of them are blocked (by a pair or an individual). To formally describe "information stability" , we define the following allocation dependent operator $H_{\mu,\mathbf{p}}$ that maps one partition profile Π to another (the meet of Π_j and $N^{(\mu,\mathbf{p},\Pi)}$ for each firm j):

$$[H_{\mu,\mathbf{p}}(\Pi)]_j(\mathbf{w}') := \Pi_j(\mathbf{w}') \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{ and all } j \in J. \tag{4}$$

A state is said to be stable if it is individually rational, not blocked, and no information can be inferred from the fact that the state is not blocked.

Definition 3. *A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is said to be **stable** if*

1. *it is individually rational,*
2. *it is not blocked, and*
3. *Π is a fixed point of $H_{\mu,\mathbf{p}}$, i.e. $H_{\mu,\mathbf{p}}(\Pi) = \Pi$.*

It is well known that in the current setup, a stable matching exists if information is complete, i.e., if $\Pi_j(\mathbf{w}') = \{\mathbf{w}'\}$ for all $j \in J$ and $\mathbf{w}' \in \Omega$, then for each $\mathbf{w}$ there exists an allocation $(\mu, \mathbf{p})$ such that the state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable (see Shapley and Shubik (1971) and Crawford and Knoer (1981)). The existence of a stable state is guaranteed by the existence of a complete-information stable state, where the particular partition Π is a fixed point of any $H_{\mu,\mathbf{p}}$ that is induced by a binary partition.

Remark 1. *Instead of a fixed point property, we can also interpret a stable state as a result of iteration. For example, suppose $(\mu, \mathbf{p})$ is a complete-information stable allocation when the type profile is $\mathbf{w}$. Given an arbitrary partition profile Π, we consider the iteration with $\Pi^0 := \Pi$ and for $k = 1, 2, \ldots$, $\Pi^k = H_{\mu,\mathbf{p}}(\Pi^{k-1})$. Then Π^∞ is a fixed point of $H_{\mu,\mathbf{p}}$, i.e., $(\mu, \mathbf{p}, \mathbf{w}, \Pi^\infty)$ is a stable state.*

3.4 Stable Allocations

Given a type assignment function $\mathbf{w}$, the payoffs of agents are actually determined by the allocation. Therefore, we concentrate on the set of allocations that can arise in stable states, which is a set $\mathscr{S}$ of allocations that are stable under some partition profile, i.e.

$$\mathscr{S}(\mathbf{w}) := \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \text{ a partition profile } \Pi \text{ s.t. } (\mu, \mathbf{p}, \mathbf{w}, \Pi) \text{ is stable}\}. \tag{5}$$

The ex ante set of stable allocations is then defined as $\mathscr{S} := \bigcup_{\mathbf{w}\in\Omega} \mathscr{S}(\mathbf{w})$. If we extract all allocations from the set of incomplete-information stable outcomes (LMPS), then it is exactly $\mathscr{S}$. Hence, our stability notion is consistent with that of LMPS in the allocation sense. See Subsection 5.2.

4 Matching Processes with Incomplete Information

Consider a job market where any pair of firm and worker can freely match themselves, and any agent can freely opt being unmatched. Suppose that agents are *myopic, i.e., once an agent or a worker-firm pair finds an opportunity to improve their status quo, they will do so by either standing alone or finding a new partner*. In this section, we will fix a realized type assignment function $\mathbf{w}^*$ and study the relationship between the set of stable allocations, $\mathscr{S}(\mathbf{w}^*)$, and blocking paths formed by a sequence of random matching. In particular, we show that with probability one an arbitrary random Learning-Blocking Path converges to an incomplete-information stable state after finitely many rematchings.

4.1 Learning-Blocking Paths

In a matching process with incomplete information, even if no rematching is observed, it is not necessarily stable. This is because such an observation of no remathcing may provide information to agents so that they can update their information. As a result, the state may change. It is possible that the updated state is blocked, which lead further to a rematching. Moreover, information updating may also occur when rematchings are observed. As a result, a matching process is associated with a *learning process* (which corresponds to a sequence of refinement of the partitional information of the agents). Thus specifying what agents know and how they update information is the first step to study a matching process. Given a state as a status quo $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ where the partition profile is common knowledge, each firm j may observe one of the following two situations:

Case 1. There is no rematching.

Case 2. There is a rematching of some blocking combination $(i, j; p)$.

In Case 1, firms commonly know that $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is not blocked, which can be distinguished from the event of the state being blocked. As a result, they update their information according to the partition $N^{(\mu,\mathbf{p},\Pi)}$ defined in Subsection 3.3, i.e. the information structure now becomes $H_{\mu,\mathbf{p}}(\Pi)$ defined by (4).

In Case 2, firms commonly observe the rematching. Additionally, they can distinguish two events: one permits $(i, j; p)$ as a blocking combination and one does not.

Denote the partition identifying the blocking combination $(i, j; p)$ by $B^{(\mu,\mathbf{p},\Pi;i,j;p)}$, i.e., $B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}') = B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}'')$ if and only if either $(i, j; p)$ blocks both $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ or $(i, j; p)$ is neither a blocking combination for one of them. Moreover, firms commonly know that firm j has observed the type of worker i. That is, j can distinguish events generated by the partition of i's type assignment. Denote the partition of i's type assignment by $O^{\{i\}}$, i.e., $O^{\{i\}}(\mathbf{w}') = O^{\{i\}}(\mathbf{w}'')$ if and only if $\mathbf{w}'(i) = \mathbf{w}''(i)$. Firms update their information according to their observations.

Formally, we say that a state $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ is *derived from another state* $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ *by satisfying a blocking combination* $(i, j; p)$ *for* $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$, which is denoted by $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ $\xleftarrow{(i,j;p)}$ $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$, if

$$
\begin{gathered}
\mu'(i) = j; \qquad \mu'((\mu)^{-1}(j)) = \emptyset; \\
\mu'(i') = \mu(i') \text{ for all } i' \in I \text{ s.t. } i' \neq (\mu)^{-1}(j) \text{ and } i' \neq i; \\
\mathbf{p}'_{i,j} = p; \qquad \mathbf{p}'_{(\mu)^{-1}(j),\emptyset} = 0; \\
\mathbf{p}'_{i',\mu'(i')} = \mathbf{p}_{i',\mu(i')} \text{ for all } i' \in I \text{ s.t. } i' \neq (\mu)^{-1}(j) \text{ and } i' \neq i,
\end{gathered}
$$

and

$$
\begin{aligned}
\Pi'_j(\mathbf{w}) =& \Pi_j(\mathbf{w}) \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}) \cap B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}) \cap O^{\{i\}}(\mathbf{w}) \text{ for all } \mathbf{w} \in \Omega; && (6) \\
\Pi'_{j'}(\mathbf{w}) =& \Pi_{j'}(\mathbf{w}) \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}) \cap B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}) \text{ for all } \mathbf{w} \in \Omega \text{ and all } j' \neq j. && (7)
\end{aligned}
$$

In words, $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ is derived from another state $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ by satisfying a blocking combination $(i, j; p)$ for $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ if (1) worker i and firm j *rematch* at the salary p; (2) the previous partners of i and j, if any, become unmatched; (3) every firm update their information according to the public observation of rematching; and (4) in addition to (3), firm j updates her information according to the additional observation of worker i's true type.

Technically, we allow one of i and j to be $\emptyset$, in which case $p = 0$. In particular, $i = \emptyset$ means that firm j dismisses her employee $\mu^{-1}(j)$; $j = \emptyset$ means that worker i resigns from her employee $\mu(i)$. Thus the operation $\xleftarrow{(i,j;p)}$ applies to both blocking pairs and individuals.

We formally describe a Learning-Blocking Path below, where μ, $\mathbf{p}$ and Π are the state variables during the matching process. At each stage, $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is the status quo.

Learning-Blocking Path

Input. An arbitrary state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$.

Initialization. Initialize μ, $\mathbf{p}$ and Π to be μ^0, $\mathbf{p}^0$ and Π^0, respectively.

Phase 1. There are two exclusive cases.

(a) $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is blocked by some individual or pair. Go to PHASE 2.

(b) $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is not blocked. Go to PHASE 3.

PHASE 2. Derive $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ such that $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi') \xleftarrow{(i,j;p)} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$.

Set $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$. Go to PHASE 1.

PHASE 3. There are two exclusive cases.

(a) $H_{\mu,\mathbf{p}}(\Pi) = \Pi$, i.e. "no rematching" provides no information. Go to END.

(b) $H_{\mu,\mathbf{p}}(\Pi) \neq \Pi$, i.e. "no rematching" contains information that could help agents refine their partitional information function.

Set $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu, \mathbf{p}, \mathbf{w}^*, H_{\mu,\mathbf{p}}(\Pi))$. Go to PHASE 1.

END. Output $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$.

We say that a LEARNING-BLOCKING PATH is *finite* if it involves finitely many rematchings, i.e. PHASE 2 and Case (b) ii of PHASE 3 are triggered finitely many times.

If $\Pi_j(\mathbf{w}) = \{\mathbf{w}\}$ for all $\mathbf{w} \in \Omega$ and $j \in J$, we are back to the complete-information environment. In this case a LEARNING-BLOCKING PATH is simply a blocking path discussed in the literature, i.e., a sequence of allocations where each allocation is derived from its immediate predecessor by satisfying one of the blocking combinations of the former (e.g. RV and Chen et al. (2010)).

A LEARNING-BLOCKING PATH may not terminate because of cycles, as in the example provided by Knuth (1976) in an ordinal preference setting. In Appendix A of the current paper, we adapt the example showing the possibility of a cycle in a transferable utility setting. However, we show that by carefully choosing blocking combinations when there are many, a LEARNING-BLOCKING PATH must terminate within finitely many rematchings.

4.2 CONVERGENCE OF LEARNING-BLOCKING PATHS

We will study two notions of path to stability, namely deterministic path to stability and random path to stability. The former notion concerns whether there exists *one* LEARNING-BLOCKING PATH that leads to a stable state; while the latter concerns the probability of reaching a stable state when blocking combinations are randomly satisfied.

4.2.1 DETERMINISTIC PATH TO STABILITY

Before stating our results, we need the following assumption.

Assumption 1. *Payments permitted in the job market are integers.*[12]

Indeed, practically payments are measured in monetary units and hence integers.[13] Given an arbitrary initial state, we show in the following theorem that by carefully choosing blocking pairs at each state, we can construct a *finite* Learning-Blocking Path (i.e., a process which consists of finitely many rematchings) that ends with a stable state.

Theorem 1. *Suppose Assumption 1 holds. Then starting from any arbitrary initial state, there exists a finite* Learning-Blocking Path *that leads to a stable state.*

Remark 2. *Obviously, every stable allocation in $\mathscr{S}(\mathbf{w}^*)$ could be reached by the* Learning-Blocking Path *starting itself. In the complete-information setting, RV observes that every stable matching can be achieved by a blocking path starting with a no-match initial status (every individual is unmatched). With incomplete information, however, the observation is not true. To see this, consider a market with one firm and one worker where the matching constitutes a stable allocation. However, the firm may not be willing to be matched with the worker concerning his counterfactual worst type.*

4.2.2 Random Path to Stability

Now we consider a random process which starts with an arbitrary state, $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, and then proceed to generate a random Learning-Blocking Path, i.e., whenever an intermediate state is blocked, randomly satisfy *a* blocking combination if there are many. In the complete information environment, RV assumes that (in our language) the probability that every particular blocking combination for a blocked allocation $(\mu, \mathbf{p})$ being chosen is bounded away from zero, which stems from a distribution over all complete-information blocking combinations and depends only on the allocation $(\mu, \mathbf{p})$ (and not on the rematching history), i.e., blocking pairs are independently drawn.

With incomplete information, similar restrictions are imposed when we study random Learning-Blocking Paths. In particular, blocking combinations are independently

[12] As we mentioned in Footnote 10, salaries must be rounded to the nearest dollar, penny, or mill. This is a technical assumption to ensure *finite* bargaining choices when a worker-firm pair negotiates, as well as, more importantly, a realistic situation in decentralized market practice which we aim to describe. See Crawford and Knoer (1981), Kelso and Crawford (1982), and Chen et al. (2016) for similar integral assumptions when *finite* matching processes are studied. We show in Appendix C that as long as the monetary unit is small enough, then there is no efficiency loss, which is consistent with LMPS. In marriage models where our results hold and there is no payment involved, of course this assumption is not necessary any more.

Moreover, one can easily construct an example, where two firms compete for one worker, the salary increment converges to zero, and the limit salary still permits a blocking combination. Therefore, finite path cannot be guaranteed even in the complete-information environment without Assumption 1.

[13] Stable states are well defined under Assumption 1 and the existence is still guaranteed (see (Crawford and Knoer, 1981, Theorem 1)). In the rest of the section, we refer stability and $\mathscr{S}(\mathbf{w}^*)$ as the ones defined under Assumption 1.

drawn from a distribution over all blocking combinations, where each combination occurs with positive probability and the probability depends only on the state but not on the rematching history. For such distribution on blocking combinations, we obtain the following result which immediately follows from Theorem 1.

Theorem 1′. *Suppose that Assumption 1 holds. Then the random* Learning-Blocking Path *starting from an arbitrary state converges to a stable state with probability one.*

Theorem 1′ predicts the ultimate state of a *random* matching market, as a corollary of Theorem 1. This random market mimics the real world situations: agents meet and negotiate randomly until they expect no more (utility or profit) improvement.

Remark 3. *In the complete-information setting, Ma (1996) considers a random order mechanism that puts agents into an empty room one by one according to a queue that is randomly drawn from all possible orders of agents with equal probability. The random order mechanism is different from RV's randomized matching mechanism only in that RV's random queue allows for pairs in the queue. However, the implications are significantly different. RV's mechanism implies that every stable matching can be achieved with positive probability, while Ma (1996) shows by example that not every stable matching can be obtained with positive probability.*[14] *Since our setting includes that of RV and thus Ma (1996) as a special case, the negative result of Ma (1996) still holds.*[15] *However, the positive result of RV fails because of the example we discussed in Remark 2.*

4.3 A Comparison of Theorem 1 and RV's Theorem

Theorem 1 parallels RV's theorem. RV constructs a sequence of subsets of agents, $\{A(l)\}_{l=1}^{r-1}$ and correspondingly a sequence of matchings $\{\mu^l\}_{l=1}^{r}$ such that at each step l, there is no blocking pair for μ^{l+1} that is contained in $A(l)$. Since blocking pairs, if any, must involve at least one agent from outside, by adding the outside agent the sequence $A(l)$ is strictly expanding. As there are only finitely many agents, the sequence will reach the set which includes everyone in the market, i.e., $A(r-1) = I \cup J$. The construction of such a sequence implies that there is no blocking pair for μ^r that is contained in $A(r-1)$, i.e., μ^r is stable. In adding an agent outside $A(l)$, the construction involves choosing the favorite partner in $A(l)$ among those who are willing to form a blocking pair with the outsider.

When firms' information is incomplete, it is no longer clear how they choose favorite partners. In particular, firms do not know which employee to favor among those who

[14] Although the result of Ma (1996) is true, the proof of one claim in his paper needs a corrigendum. See Klaus and Klijn (2007a).

[15] Although the counterexample in Ma (1996) assumes non-transferable utility, transfer actually brings no obstacle to adopt his counterexample in the current setting. This is because we can keep the preference order while make all matching surpluses smaller than 1, the monetary unit set by Assumption 1, such that no payment exists in any individually rational state.

are willing to form a blocking pair, as they may not know these workers' types. More precisely, a firm may not be willing to form a blocking pair with her *de facto* favorite worker in the worry of his worst possible type which she considers possible.

The property that no blocking pair for μ^{l+1} is contained in $A(l)$ plays a key role in the proof of their theorem. With incomplete information, firms in $A(l)$ observe either that there is a rematching or that there is no rematching, both of which may lead to information updating. This implies that there may be blocking pairs for μ^{l+1} that is contained in $A(l)$ when μ^{l+2} is derived. As a result, RV's argument does not apply here.

In view of these issues, our proof proceeds with two key ideas, which are presented in Lemma 1 and Lemma 2 respectively.

4.3.1 Best Partner

First, instead of finding the "best" partner, we introduce a "lost mate finding rule" , which solve the problem of finding the best partner into small steps. The "lost mate finding rule" is applicable to both complete- and incomplete-information environments.

Lemma 1. *Fix a state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, a subset of agents $A \subset I \cup J$ whose partners under μ^0 are all in A and a worker $i^0 \notin A$ (resp. a firm $j^0 \notin A$). Suppose every matched agent in A has a positive payoff and there is a blocking pair where the blocking worker is i^0 (resp. the blocking firm is j^0) and the blocking firm (worker) is in A. Then there exists a state $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ and a finite* Learning-Blocking Path *starting with $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and ending with $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ such that the following two statements are true.*

1. *There is no blocking pair that is contained in $A \cup \{i^0\}$ and involves i^0 (resp. j^0) for the state $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$.*
2. *Compared with $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ makes the payoffs of all firms (resp. workers') in A unchanged except for one firm (resp. worker) who gets strictly higher payoff.*

Since the two cases in Lemma 1 are symmetric, we focus on the case with i^0. The Learning-Blocking Path for Lemma 1 is constructed by ALGORITHM 1 below. Along the process, μ, $\mathbf{p}$, Π, $\bar{\imath}$, $\bar{\jmath}$ and $\bar{p}$ are state variables, where $(\bar{\imath}, \bar{\jmath}; \bar{p})$ is the match to be restored by what we call "lost mate finding rule" . The "lost mate finding rule" used in Process is illustrated in Table 1 where payments are ignored for simplicity. Generally, suppose $(\mu'', \mathbf{p}'')$ is derived by satisfying a blocking pair (i', j') of $(\mu', \mathbf{p}')$, which in turn is derived by satisfying a blocking pair (i, j) of $(\mu, \mathbf{p})$. If $i' = i$ and $\mu^{-1}(j) \neq \emptyset$, then $(\mu^{-1}(j), j; \mathbf{p}_{\mu^{-1}(j),j})$ blocks $(\mu'', \mathbf{p}'')$ when their blocking payoffs are positive. If $j' = j$ and $\mu(i) \neq \emptyset$, then $(i, \mu(i);\ \mathbf{p}_{i,\mu(i)})$ blocks $(\mu'', \mathbf{p}'')$ when their blocking payoffs are positive.

Table 1: An illustration of the “lost mate finding rule” .

Matchings	Matching Details					Blocking Pairs
μ	$\dots$	i	$\mu^{-1}(j)$	$\dots$	$\dots$	
	$\dots$	$\dots$	j	j'	$\dots$	(i, j)
μ'	$\dots$	i	$\mu^{-1}(j)$	$\dots$	$\dots$	
	$\dots$	j	$\emptyset$	j'	$\dots$	(i, j')
μ''	$\dots$	i	$\mu^{-1}(j)$	$\emptyset$	$\dots$	
	$\dots$	j'	$\emptyset$	j	$\dots$	$(\mu^{-1}(j), j)$

Algorithm 1

Input. A state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and a worker i^0.

Initialization. Initialize μ, $\mathbf{p}$ and Π to be μ^0, $\mathbf{p}^0$ and Π^0 respectively. Initialize $\bar{\imath}$, $\bar{\jmath}$, and $\bar{p}$ to be the dummy agent $\emptyset$, an arbitrary firm and an arbitrary real number respectively.

Process. Consider two mutually exclusive cases.

(a) If there exists no blocking combination $(i, j; p)$ for $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ such that $\{i, j\} \subset A \cup \{i^0\}$ and $i = i^0$, go to End.

(b) Otherwise, arbitrarily pick a blocking combination $(i, j; p)$ for $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ such that $\{i, j\} \subset A \cup \{i^0\}$ and $i = i^0$. Derive $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ such that $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi') \xleftarrow{(i,j;p)} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$. Consider two mutually exclusive cases.

i. If $\bar{\imath} \in I$, derive $(\mu'', \mathbf{p}'', \mathbf{w}^*, \Pi')$ such that $(\mu'', \mathbf{p}'', \mathbf{w}^*, \Pi') \xleftarrow{(\bar{\imath},\bar{\jmath};\bar{p})} (\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$, i.e., let $\bar{\jmath}$ “find her lost mate” at the previous salary. Set $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu'', \mathbf{p}'', \mathbf{w}^*, \Pi')$.

ii. If $\bar{\imath} \notin I$, set $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$.

Set $\bar{\imath}$ to be $\mu^{-1}(j)$, $\bar{\jmath}$ to be j and $\bar{p}$ to be $\mathbf{p}_{\bar{\imath},\bar{\jmath}}$. Go to Process.

End. Output $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ to be $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$.

Proof of Lemma 1. We prove the statements with input i^0.

1. We first claim that in Process, $(\bar{\imath}, \bar{\jmath}; \bar{p})$ is a blocking combination for $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ if $\bar{\imath} \in I$. This is because $\bar{\imath}$ and $\bar{\jmath}$ are both unmatched while the matched payoffs are positive. Therefore, Algorithm 1 produces a Learning-Blocking Path. To see finiteness of the Learning-Blocking Path, note that whenever we trigger Process, worker i^0's payoff strictly increases. Since we assumed the finiteness of match surplus and integer payments, worker i^0 can be made better off with only finitely many rematchings.

2. We index PROCESS by $k = 1, 2, \ldots$. The assumption of the lemma implies that $k \geq 2$. For $k \geq 2$, when PROCESS k is triggered, we argue that the payoffs of all firms in set A are unchanged except for the blocking firm j who gets strictly higher payoff compared with the payoffs under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. Therefore, Statement 2 of the lemma follows by induction in k. More precisely, in each PROCESS satisfying $(i, j; p)$, where $i = i^0$, may change the payoffs of at most two firms, i.e., j and $\mu(i^0)$. The payoff of firm j gets higher. If $\mu(i^0) \notin A$, we are done. If $\mu(i^0) \in A$, then $\mu(i^0) = \bar{j}$, who will "find her lost mate" and get back her payoff under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. □

REMARK 4. *Lemma 1 enables an alternative proof for RV's theorem if there is no incomplete information and transfer, also an alternative proof of (Chen et al., 2010, Theorem 1) if information is complete.* ALGORITHM 1 *ensures the flexibility when we choose blocking combinations, so that the proof does not rely on the "favorite mate picking rule" used in RV's proof, or the "initiator getting the lion's share of the resulting surplus" rule introduced by Chen et al. (2010) (also used in the algorithm of Chen et al. (2016)).*

4.3.2 IDENTIFYING INFORMATION UPDATING

The second idea, unlike the first one, is specific to the incomplete-information environment. Formally, we say that a set of agents A *contains no internal blocking pair under state* $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ if no pair or individual in A blocks $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$. Consider a set of agents that contains no internal blocking pair under a state. We can construct a finite LEARNING-BLOCKING PATH that either (1) leads to a larger set with no internal blocking pair at the updated state or (2) ensures that at least one firm is matched with a worker which she has never met before. The former case is in line with the proof of RV, whereas the latter case triggers irreversible information updating and thus eliminates possible cycles in the LEARNING-BLOCKING PATH.

LEMMA 2. *Fix a state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and a subset of agents $A^0 \subset I \cup J$ whose partners under μ^0 are all in A. Suppose A^0 contains no internal blocking pair under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. and every matched agent in A^0 has a positive payoff. If $(i^0, j^0; p^0)$ is a blocking combination for $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ satisfying $j^0 \in A^0$ and $i^0 \notin A^0$ (resp. $i^0 \in A^0$ and $j^0 \notin A^0$), then there exists a finite* LEARNING-BLOCKING PATH *starting with $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and A^0, and ending with $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ and another subset A^1 of $I \cup J$ such that one of the following two statements is true.*

1. *$A^1 := A^0 \cup \{i^0\}$ (resp. $A^0 \cup \{j^0\}$) contains no internal blocking pair under $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$.*
2. *Under $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$, there exists a firm $\hat{\jmath}$ who learns the type of a worker which she did not learn at the initial state, i.e., $|\{\mathbf{w}((\mu^1)^{-1}(\hat{\jmath})) : \mathbf{w} \in \Pi^0_{\hat{\jmath}}(\mathbf{w}^*)\}| \geq 2$ but $|\{\mathbf{w}((\mu^1)^{-1}(\hat{\jmath})) : \mathbf{w} \in \Pi^1_{\hat{\jmath}}(\mathbf{w}^*)\}| = 1$.*

Lemma 2 helps us construct the LEARNING-BLOCKING PATH towards stability from an arbitrary initial state. The main idea is to adjust the allocation so that $A^1 := A^0 \cup \{i^0\}$ contains no internal blocking pair. To do this, we (apply Lemma 1 to) match i^0 with his best partner in A, match the initial partner α^0 (if any) of the best partner of i^0 to α^0's best partner in A, match the initial partner α^1 (if any) of the best partner of α^0 to α^1's best partner in A, and so on. Then, at some step t, we have either (1) the best partner of agent α^t has no initial partner or the initial partner prefers standing alone after α^t is matched to her best partner (i.e., the initial partner of α^t's best partner ends up being unmatched); or (2) there is a blocking pair in A^1 with a worker i not being the initial partner of α^t's best partner. Clearly, if (1) happens and (2) does not, then A^1 contains no blocking pair. The main argument in Lemma 2 shows that whenever (2) happens, the rematching partner of i must have a nontrivial belief updating.

We prove Lemma 2 by construction. The LEARNING-BLOCKING PATH for the proof of Lemma 2 is constructed by the following algorithm. Apart from μ, $\mathbf{p}$ and Π, we also keep track of the state variable α which corresponds to α^0, α^1, α^2,..., illustrated above and will be updated during the matching process.

ALGORITHM 2

INPUT. A state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and a subset A^0 of $I \cup J$ that contains no internal blocking pair under $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$.

INITIALIZATION. Initialize μ, $\mathbf{p}$, Π and A to be μ^0, $\mathbf{p}^0$, Π^0 and A^0 respectively. Set α to be i^0 as the outside worker of the blocking pair.

PROCESS. Set A to be $A^0 \cup \{i^0\}$. Run ALGORITHM 1, with input $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ and α. Denote the output by $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ and $(\mu^0)^{-1}(\mu'(\alpha))$ by $\bar{i}$. Set $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$. Consider three mutually exclusive cases.

(a) If A contains no blocking pair for $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$, go to END.

(b) If there exists a blocking combination $(i, j; p)$ for $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ such that $\{i, j\} \subset A$ and $i = \bar{i}$, then go to PROCESS with $\alpha = \bar{i}$.

(c) Otherwise, arbitrarily pick a blocking combination $(i, j; p)$ such that $\{i, j\} \subset A$. Derive $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ such that $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi') \xleftarrow{(i,j;p)} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$. Set A to be $\varnothing$ and $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$. Go to END.

END. Set the output state $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ as $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ and A^1 as A.

As before, a LEARNING-BLOCKING PATH is finite if it involves finitely many rematchings.

Proof of Lemma 2. We first prove Lemma 2 for the case with $j^0 \in A^0$ and $i^0 \notin A^0$. PROCESSes are indexed by $t = 1, 2 \ldots$, and denote by α^t the value assigned to the state variable α during PROCESS t.

We proceed to show that the LEARNING-BLOCKING PATH produced by ALGORITHM 2 involves finitely many rematchings. By Lemma 1, each time we run ALGORITHM 1, all firms' payoffs are unchanged except for one firm who gets strictly higher payoff. Since the surplus from each match is a real number (thus finite) and only integer payments are permitted (Assumption 1), we know that for each agent higher payoff is permitted only finitely many times. Note in addition that $I \cup J$ is finite, PROCESS can be triggered only finitely many times. Again by Lemma 1, ALGORITHM 1 produces a finite LEARNING-BLOCKING PATH, which implies that ALGORITHM 2 also produces a finite LEARNING-BLOCKING PATH. Let n be the largest index of PROCESS, i.e., $t = 1, 2, \ldots, n$.

If $A^1 \neq \varnothing$, then A^1 contains no internal blocking pair under $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$ by construction. Hence, it suffices to show that when $A^1 = \varnothing$, under $(\mu^1, \mathbf{p}^1, \mathbf{w}^*, \Pi^1)$, there exists a firm $\hat{\jmath}$ such that $|\{\mathbf{w}(\mu^{-1}(\hat{\jmath})) : \mathbf{w} \in \Pi^0_{\hat{\jmath}}(\mathbf{w}^*)\}| \geq 2$ but $|\{\mathbf{w}(\mu^{-1}(\hat{\jmath})) : \mathbf{w} \in \Pi^1_{\hat{\jmath}}(\mathbf{w}^*)\}| = 1$. Observe that $A^1 = \varnothing$ results only under Case (c) of PROCESS. We consider the blocking combination $(i, j; p)$ at PROCESS n.

Suppose to the contrast that $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi^0_j(\mathbf{w}^*)\}| = 1$, i.e., for every $\mathbf{w} \in \Pi^0_j(\mathbf{w}^*)$, $\mathbf{w}(i) = \mathbf{w}^*(i)$. Then j has complete information about i's type at state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. We distinguish two cases. First, suppose that i has never been α at any previous PROCESS, then i's employer keeps unchanged since the start of the algorithm. Since all firms' payoffs get weakly higher whenever we run ALGORITHM 1, j's payoff was worse at state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ than at the beginning of Case (c) in PROCESS n. Since $(i, j; p)$ is a blocking pair for the status quo and j has complete information about i's type at state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, we know that $(i, j; p)$ must be a blocking combination for the state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$. But $\{i, j\}$ was contained in A, a contradiction.

Second, suppose that i has been α^t at some pervious PROCESS t. Without loss of generality, let t be the largest index with $\alpha^t = i$. After we run ALGORITHM 1 in PROCESS t, there was no further blocking pair that involves i by Lemma 1. Particularly, $(i, j; p)$ was not a blocking combination. By hypothesis, j has complete information about i's type at state $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$ and thus right after we run ALGORITHM 1 in PROCESS t. Since all firms' payoffs get weakly higher whenever we run ALGORITHM 1 in subsequent PROCESSes, firm j's payoff right after we run ALGORITHM 1 in PROCESS t was worse than at the beginning of Case (c) in PROCESS n. Since $(i, j; p)$ is a blocking combination for the status quo and j has complete information about i's type, we know that $(i, j; p)$ must be a blocking combination for the state right after we run ALGORITHM 1 in PROCESS t, a contradiction. Therefore, $|\{\mathbf{w}(i) : \mathbf{w} \in \Pi^0_j(\mathbf{w}^*)\}| \geq 2$. Let $\hat{\jmath}$ be j and, equivalently, $(\mu^1)^{-1}(\hat{\jmath}) = i$. Then $|\{\mathbf{w}((\mu^1)^{-1}(\hat{\jmath})) : \mathbf{w} \in \Pi^0_{\hat{\jmath}}(\mathbf{w}^*)\}| \geq 2$ but $|\{\mathbf{w}((\mu^1)^{-1}(\hat{\jmath})) : \mathbf{w} \in \Pi^1_{\hat{\jmath}}(\mathbf{w}^*)\}| = 1$.

For the case with $j^0 \notin A^0$ and $i^0 \in A^0$, ALGORITHM 2 is defined analogously by interchanging workers and firms. The statement of the lemma is straightforward when $A^1 \neq \varnothing$. To show the statement when $A^1 = \varnothing$, we consider the blocking combination $(i, j; p)$ at the latest PROCESS before END. Assuming that j had complete information about i's type at $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$, we can derive contradictions by symmetric arguments as in the case with $j^0 \in A^0$ and $i^0 \notin A^0$. Finally, the LEARNING-BLOCKING PATH is finite for symmetric reasons as in the case with $j^0 \in A^0$ and $i^0 \notin A^0$. □

4.4 PROOF OF THEOREM 1

To prove Theorem 1, we construct a specific LEARNING-BLOCKING PATH by applying a path constructing algorithm which we describe below. Like RV's algorithm, our algorithm also produces a sequence of subsets of agents which contains no internal blocking pairs. By Lemma 2, the subsets may either expand or shrink to an empty set along the sequence. We have argued that the later case implies nontrivial information updating, i.e., at least one firm is matched with a worker whom she has never matched with. Along the sequence, the subsets may also shrink because of the observation of no rematching, which again implies nontrivial information updating as we will show below. Since we have finite agents, this will be enough to ensure that the sequence of subsets containing no internal blocking pair will ultimately expand to include all agents.

Without loss of generality, we assume that the initial state is individually rational. Otherwise, finitely many breaking-ups will lead to an individually rational state. The LEARNING-BLOCKING PATH is constructed by the following algorithm. Apart from μ, $\mathbf{p}$, and Π, during the matching process A is another state variable, which represents a set of agents containing no blocking pair.

PATH CONSTRUCTING ALGORITHM

INPUT. An arbitrary individually rational state: $(\mu^0, \mathbf{p}^0, \mathbf{w}^*, \Pi^0)$.

INITIALIZATION. Initialize μ, $\mathbf{p}$, Π and A to be μ^0, $\mathbf{p}^0$, Π^0 and A^0 respectively.

PHASE 1. There are two mutually exclusive cases.

(a) $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is blocked. Go to PHASE 2.

(b) $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is not blocked. Go to PHASE 3.

PHASE 2. There are two mutually exclusive cases.

(a) For any blocking combination $(i, j; p)$ of the current state, $i \notin A$ and $j \notin A$

Pick a blocking combination $(i, j; p)$. Derive A' such that $A' = A \cup \{j\}$. Set A to be A'. Go to PHASE 2.

(b) Otherwise, run ALGORITHM 2 for some blocking combination with $i \in A$ or $j \in A$, which will output both A and $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$. Go to PHASE 1.

PHASE 3. There are two mutually exclusive cases.

(a) If $H_{\mu,\mathbf{p}}(\Pi) = \Pi$, go to END.

(b) If $H_{\mu,\mathbf{p}}(\Pi) \neq \Pi$, derive Π' such that $\Pi' = H_{\mu,\mathbf{p}}(\Pi)$. Set $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi')$.

i. If $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is not blocked, then go to PHASE 3.

ii. If $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ is blocked by some $(i, j; p)$, then derive $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$ such that $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi') \xleftarrow{(i,j;p)} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$. Set A to be $\varnothing$ and $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ to be $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi')$. Go to PHASE 1.

END. Output $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$.

We proceed to show that the PATH CONSTRUCTING ALGORITHM terminates after finitely many rematchings. Obviously whenever PHASE 1 and PHASE 2 start, the set A contains no blocking pair for the status quo. Since matched agents in A are all from satisfying blocking combinations, their payoffs are all positive, which means that the conditions for Lemma 2 are satisfied whenever Case (b) of PHASE 2 is triggered.

Proof of Theorem 1. We first claim that the set A is set to be empty set at most finitely many times. The set A is set to be empty set only when one of the following two cases is triggered: Subcase (b) of PHASE 2 or Subcase (b) ii of PHASE 3. By Lemma 2, A can be empty in Subcase (b) of PHASE 2 at most $|I| \times |J|$ many times. Note that when Subcase (b) ii of PHASE 3 starts, the partition profile Π' is strictly finer Π. Since Ω and J are both finite, the partition profile can be finer only finitely many times, i.e., Subcase (b) ii of PHASE 3 can be triggered at most finitely many times.

Second, the PATH CONSTRUCTING ALGORITHM stops in finite time. To see this, note that the set A can either decrease (to empty set) or increase. It decreases only finitely many times by the above argument. In each case of PHASE 2, set A is strictly increasing. The increasing direction is finite because of the finiteness of $I \cup J$ and the finiteness of the decreasing direction. Therefore, both the increasing direction and the decreasing direction are finite, which implies that the PATH CONSTRUCTING ALGORITHM stops in finite time.

Finally, we argue that the output state of the PATH CONSTRUCTING ALGORITHM is stable. Since the PATH CONSTRUCTING ALGORITHM ends only if PHASE 3 is triggered, which in turn only if Case (b) of PHASE 1 or Case (b) i of PHASE 3 is triggered, we know that the output is not blocked. The ending condition in Case (a) of PHASE 3 implies that the output is a stable state. □

REMARK 5. *For simplicity, the above proof depends both on the number of type assignments being finite and on the number of agents being finite. Actually, we can make the proof independent of the finiteness of* Ω *by arguing that when Subcase (b) ii of* PHASE 3 *starts, the blocking firm* j *has never been matched with worker* i*. As a result, Subcase (b) ii of* PHASE 3 *can be triggered at most* $|I| \times |J|$ *times.*

To see the claim, suppose j *has been matched with* i *before the underlying* PHASE 3*. Then* j *has complete information about* i*'s type. That is, for every* $\mathbf{w} \in \Pi_j(\mathbf{w}^*)$, $\mathbf{w}(i) = \mathbf{w}^*(i)$*. Since* $(i, j; p)$ *blocks the status quo,* $(i, j; p)$ *must also block the state when the underlying* PHASE 3 *starts, which contradicts the triggering condition of* PHASE 3*, i.e. no rematching. Therefore,* j *has never been matched with* i*.*

5 DISCUSSIONS

5.1 ROBUSTNESS OF CONVERGENCE

The LEARNING-BLOCKING PATH described in Subsection 4.1 includes three kinds of inference that can be drawn from different observations. However, the firms may not be as sophisticated as we modeled. For example, they may not be able to keep track of the partition profile Π which represents the information structure of the entire market. It is also possible that firms do not have perfect recall as we implicitly assumed in Subsection 4.1. Alternatively, they may be more sophisticated in coming up with arguments to rule out more type profiles than we would like them to.

In general, we can think of an *information updating pattern* as three mappings $H_{\mu,\mathbf{p}}$, $B_{\mu,\mathbf{p},\Pi;i,j,p}$ and $P_{\mu,\mathbf{p},\Pi;i,j,p}$ that specify agents' information updating for each type of observations: $H_{\mu,\mathbf{p}} : \Pi \longmapsto \Pi'$ is defined by (4) for firms under the case of no rematching, $P_{\mu,\mathbf{p},\Pi;i,j,p} : \Pi_j \longmapsto \Pi'_j$ is for j under the case of rematching of $(i, j; p)$, and $B_{\mu,\mathbf{p},\Pi;i,j,p} : \Pi_{j'} \longmapsto \Pi'_{j'}$ is for firm $j' \neq j$ under the case of rematching by $(i, j; p)$.

When firms are not as sophisticated as we have modeled, $P_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_j)$ and $B_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_{j'})$ are coarser than those updated partitions defined by (6)-(7). Theorem 1 holds if we use a coarser information updating pattern along the LEARNING-BLOCKING PATH, as long as $P_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_j)$ and $B_{\mu,\mathbf{p},\Pi;i,j,p}(\Pi_{j'})$ are finer than Π'_j and $\Pi'_{j'}$, respectively, where

$$\begin{aligned} \Pi'_j(\mathbf{w}) =& \Pi_j(\mathbf{w}) \cap O^{\{i\}}(\mathbf{w}) \text{ for all } \mathbf{w} \in \Omega; \\ \Pi'_{j'}(\mathbf{w}) =& \Pi_{j'}(\mathbf{w}) \qquad\qquad \text{for all } \mathbf{w} \in \Omega \text{ and all } j' \neq j. \end{aligned}$$

Actually we allow for any temporary change to the partition profile, which may be coarser or finer, as long as the change happens only finitely many times.

Firms may also be more sophisticated than we have modeled. For example, when myopic agents can observe not only the allocation change but also firms' offer making and workers' response, they may experience some intermediate stage between two allocations. During those intermediate stages, firms may have the access to more information. In this case, one can enrich an information updating pattern by considering more possible observations. Reformulating the convergence problem under the alternative information updating patterns may be an interesting extension of our analysis.

5.2 Equivalence in Stability Notions

The stability notion of LMPS is *ex ante* in that it is independent of the true type assignment function. One can imagine that there is an outside analyst who knows the model except for $\mathbf{w}^*$, and wants to predict possible outcomes for the market. The notion of blocking is designed to only exclude outcomes that they can be certain to be "blocked" . Formally, an *(ex post) outcome* $(\mu, \mathbf{p}, \mathbf{w})$ specifies an allocation and a type assignment function. The individual rationality of an ex post outcome is defined in a same manner as Definition 1. Let Σ be a set of ex post outcomes.

Definition 4. *(LMPS) Fix a nonempty set of individually rational outcomes, Σ. A matching outcome $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma$ is Σ-**blocked** if there exists a worker-firm pair (i, j) and a payment $p \in \mathbb{R}$ satisfying*

$$\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \quad \textit{and} \tag{8}$$

$$\phi_{\mathbf{w}'(i),\mathbf{f}(j)} - p > \phi_{\mathbf{w}'(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \tag{9}$$

for all $\mathbf{w}' \in \Omega$ satisfying

$$(\mu, \mathbf{p}, \mathbf{w}') \in \Sigma \tag{10}$$

$$\mathbf{w}'(\mu^{-1}(j)) = \mathbf{w}(\mu^{-1}(j)) \tag{11}$$

$$\nu_{\mathbf{w}'(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}. \tag{12}$$

*A matching outcome $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma$ is Σ-**stable** if it is not Σ-blocked.*

Inequalities (8) and (9) state two better-off conditions. Conditions (10-12) mean that firm j considers only "reasonable" type assignment functions, which are consistent with i) the outcome set Σ, ii) her "observation" $\mathbf{w}(\mu^{-1}(j))$ and iii) worker i's willingness (12) to be matched with j at p. In other words, blocking conditions in Definition 4 says that *were* $\mathbf{w}$ the true type assignment function, then $(\mu, \mathbf{p}, \mathbf{w})$ would be blocked based on the information of Σ, i.e. only outcomes in Σ are possible.

The set of outcomes that are immune to the blocking described above is given by the below iteration process. Let Σ^0 be the set of all individually rational outcomes. For $k \geq 1$, define

$$\Sigma^k := \left\{(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^{k-1} : (\mu, \mathbf{p}, \mathbf{w}) \textit{ is } \Sigma^{k-1} - \textit{stable}\right\}. \tag{13}$$

The set of *incomplete-information stable outcomes* (LMPS) is given by $\Sigma^\infty := \cap_{k=1}^{\infty} \Sigma^k$.

The following theorem shows the connection between our stability notion and that of LMPS, whose proof can be found in Appendix B.

Theorem 2. *$(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$ if and only if there exists a partition profile Π such that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable.*

Recall that $\mathscr{S}$ is the ex ante set of stable allocations defined in Subsection 4.3. By definition of $\mathscr{S}$, we have

$$\begin{aligned}
\mathscr{S} &= \bigcup_{\mathbf{w} \in \Omega} \mathscr{S}(\mathbf{w}) \\
&= \bigcup_{\mathbf{w} \in \Omega} \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \text{ a partition profile } \Pi \text{ s.t. } (\mu, \mathbf{p}, \mathbf{w}, \Pi) \text{ is stable } \} \\
&= \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \mathbf{w} \in \Omega \text{ and a partition profile } \Pi \text{ s.t. } (\mu, \mathbf{p}, \mathbf{w}, \Pi) \text{ is stable } \}.
\end{aligned}$$

The following corollary is an immediate result of Theorem 2, which says that our notion of stability (Definition 1-3) is consistent with that of LMPS in terms of allocation.

Corollary 1. $\mathscr{S} = \{(\mu, \mathbf{p}) \in \mathscr{A} : \exists \mathbf{w} \in \Omega \textit{ s.t. } (\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty\}$.

Remark 6. *Our notion of stability coincides with the notion proposed LMPS when the only source of firms' heterogeneous information is that each firm can observe the type of her own employee. To make a precise comparison, consider a stable outcome $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$. Let Π^0 be the partition profile over Ω that distinguishes for each firm only the type of her employee (if any), i.e., firm j puts all type assignment functions $\mathbf{w}' \in \Omega$ with the same $\mathbf{w}'(\mu^{-1}(j))$ into the same partition cell. We consider the iteration starting with Π^0 and for $k = 1, 2, \ldots$, $\Pi^k := H_{\mu,\mathbf{p}}(\Pi^{k-1})$. The intuition is that $(\mu, \mathbf{p}, \mathbf{w}, \Pi^\infty)$ must be a stable state in the sense of Definition 3, which serves as our construction of a stable state for the necessity direction of Theorem 2.*[16]

Since all complete-information stable outcomes are included in Σ^∞, we can compare the above iteration with the one in Remark 1 for those outcomes. A main difference between two iterations, and thus two stability notions, is that we allow for arbitrary starting partition profile Π^0 in Remark 1. For outcomes that are in Σ^∞ but not complete-information stable, our stability notion still allows for flexible Π to make $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ stable, instead of a particular Π^∞. Our flexible formulation of firms' information facilitates the study of matching processes.

[16] Detailed argument is included in Appendix B.

5.3 Stability and Efficiency

As has been shown in LMPS, under the following intuitive assumptions every incomplete-information stable allocation is efficient (in the sense of maximizing total surplus).

Assumption 2. *(Monotonicity) The worker premuneration values ν_{wf} and firm premuneration values ϕ_{wf} are increasing in w and f, with ν_{wf} strictly increasing in w and ϕ_{wf} strictly increasing in f.*

Assumption 3. *(Supermodularity) The worker premuneration value ν_{wf} and the match surplus $\nu_{wf} + \phi_{wf}$ are strictly supermodular in w and f.*

In a market where there is a unit of measurement for payments, does the random matching process converge to an efficient allocation, provided the above assumptions? We show that this is true as long as the numeraire is sufficiently small.

Theorem 3. *Under Assumptions 1-3, every stable allocation is efficient if the monetary unit is sufficiently small.*

The rigorous statement and its proof are provided in Appendix C.

6 Concluding Remarks

In this paper, we introduce a notion of stability for matching with one-sided incomplete information which accommodates firm-specific information. Moreover, we show the convergence of random Learning-Blocking Paths to stable states, which extend the result on paths to stability due to Roth and Vande Vate (1990) to an incomplete-information environment. In proving the result, we provide a new proof for RV's theorem. By applying the "lost mate finding rule" , the proof avoids the intermediate optimization problems. It makes the proof more elementary, as we illustrated in Subsection 4.3. Furthermore, it is crucial to describe how firms form and update their possibilistic information, and how they utilize the information to draw inference. From this angle, our analysis provides a benchmark to study the strategic foundations of stability in a decentralized matching market such as Lauermann and Nöldeke (2014), which LMPS leaves as an open question.[17]

[17]"...Under incomplete information, the connection between stable matches and the process by which stable matches are formed is yet less obvious. In the process of encountering others and accepting or rejecting matches, the agents are likely to *learn* about their environment. As a result, the information structure prevailing at the end of the matching process will typically differ from that at the beginning..." (LMPS, pp. 570)

"...agents *make inferences* from intermediate outcomes during the matching process, so the set of possible incomplete-information stable outcomes becomes a 'moving target.' Providing decentralized foundations for both complete- and incomplete-information stable matchings is an open and obviously interesting problem." (LMPS, pp. 543)

Appendices

A An Example: Learning and Cycle Possibility

A Learning-Blocking Path with allocations $(\mu^1, \mathbf{p}^1), \ldots, (\mu^r, \mathbf{p}^r), \ldots$ is a *cycle* if there is a finite r satisfying that for any l $(\mu^{l+r}, \mathbf{p}^{l+r}) = (\mu^l, \mathbf{p}^l)$ and there is no belief updating after $(\mu^r, \mathbf{p}^r)$. The following example illustrates one step of learning and shows the possibility of cycles.

Example 1. *There are three workers, $I = \{x, y, z\}$, and three firms, $J = \{a, b, c\}$. Firms' types are, respectively, f_1, f_2, f_3. The set of possible worker type assignment functions is $T = \{\mathbf{w}^*, \mathbf{w}\} := \{(w_1, w_2, w_3), (w_4, w_5, w_3)\}$, where vectors are used for convenience, i.e. the arguments for each vector are the types of worker x, y, and z, respectively. The realized type profile is $\mathbf{w}^* = (w_1, w_2, w_3)$. The premuneration values for workers and firms with each possibility of type combination are given as follows.*

$$\begin{array}{lllll}
\phi_{w_1,f_1} = 2 & \phi_{w_2,f_1} = 2 & \phi_{w_3,f_1} = 2 & \phi_{w_4,f_1} = 2 & \phi_{w_5,f_1} = 0 \\
\phi_{w_1,f_2} = 3 & \phi_{w_2,f_2} = 3 & \phi_{w_3,f_2} = 3 & \phi_{w_4,f_2} = 4 & \phi_{w_5,f_2} = 3 \\
\phi_{w_1,f_3} = 0 & \phi_{w_2,f_3} = 10 & \phi_{w_3,f_3} = 2 & \phi_{w_4,f_3} = 0 & \phi_{w_5,f_3} = 0
\end{array}$$

$$\begin{array}{lllll}
\nu_{w_1,f_1} = 2 & \nu_{w_2,f_1} = 2 & \nu_{w_3,f_1} = 2 & \nu_{w_4,f_1} = 2 & \nu_{w_5,f_1} = 0 \\
\nu_{w_1,f_2} = 2 & \nu_{w_2,f_2} = 2 & \nu_{w_3,f_2} = 2 & \nu_{w_4,f_2} = 3 & \nu_{w_5,f_2} = 2 \\
\nu_{w_1,f_3} = 0 & \nu_{w_2,f_3} = 2 & \nu_{w_3,f_3} = 2 & \nu_{w_4,f_3} = 0 & \nu_{w_5,f_3} = 2
\end{array}$$

Table 2 shows a step of learning and a cycle.

Since firm a and b face no uncertainty given the starting allocation $(\mu^1, \mathbf{p}^1)$, we specify only firm c's belief and omit the history if there is no ambiguity. The second row for every allocation stands for the payments.

In the above example, if the blocking pair for $(\mu^4, \mathbf{p}^4)$ were (x, a) and the blocking salary is 0, then we will have the following allocation, which is complete-information stable. Thus a blocking path does not necessarily stop with a stable allocation, while we can make it true by picking blocking combinations, say, $(x, a; 0)$ for $(\mu^4, \mathbf{p}^4)$, carefully.

$$(\mu, \mathbf{p}) = \begin{array}{cccc}
a & b & c & \text{(firms)} \\
f_1 & f_2 & f_3 & \text{(firm type assignment)} \\
2 & 3 & 6 & \text{(firms' payoff)} \\
0 & 0 & 4 & \text{(payment scheme)} \\
2 & 2 & 6 & \text{(workers' payoff)} \\
w_1 & w_3 & w_2 & \text{(worker type assignment)} \\
x & z & y & \text{(workers)}
\end{array}$$

Table 2: Learning and Cycle in a Blocking Path

Allocations	Beliefs	Blocking Combinations
$(\mu^1, \mathbf{p}^1) = \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ x & y & z \end{array}$	$\Pi_c = \{\{\mathbf{w}^*, \mathbf{w}\}\}$	
$(\mu^1, \mathbf{p}^1) = \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ x & y & z \end{array}$	$\Pi_c = \{\{\mathbf{w}^*\}, \{\mathbf{w}\}\}$	(y, c) at $p^1 = 4$
$(\mu^2, \mathbf{p}^2) = \begin{array}{cccc} a & b & c & \\ 0 & & 4 & \\ x & & y & z \end{array}$	$\Pi_c = \{\{\mathbf{w}^*\}, \{\mathbf{w}\}\}$	(x, b) at $p^2 = 1$
$(\mu^3, \mathbf{p}^3) = \begin{array}{cccc} a & b & c & \\ & 1 & 4 & \\ & x & y & z \end{array}$	$\Pi_c = \{\{\mathbf{w}^*\}, \{\mathbf{w}\}\}$	(z, b) at $p^3 = 0$
$(\mu^4, \mathbf{p}^4) = \begin{array}{cccc} a & b & c & \\ & 0 & 4 & \\ & z & y & x \end{array}$	$\Pi_c = \{\{\mathbf{w}^*\}, \{\mathbf{w}\}\}$	(z, a) at $p^4 = 1$
$(\mu^5, \mathbf{p}^5) = \begin{array}{cccc} a & b & c & \\ 1 & & 4 & \\ z & & y & x \end{array}$	$\Pi_c = \{\{\mathbf{w}^*\}, \{\mathbf{w}\}\}$	(x, a) at $p^5 = 0$
$(\mu^6, \mathbf{p}^6) = \begin{array}{cccc} a & b & c & \\ 0 & & 4 & \\ x & & y & z \end{array}$	$\Pi_c = \{\{\mathbf{w}^*\}, \{\mathbf{w}\}\}$	(x, b) at $p^6 = 1$

B Proof of Theorem 2

Necessity. Suppose $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^\infty$. Denote the partition over Ω that indicates whether $(\mu, \mathbf{p}, \mathbf{w}') \in \Sigma^\infty$ or not by $T_{\mu,\mathbf{p}}$, i.e., $T_{\mu,\mathbf{p}}(\mathbf{w}') = T_{\mu,\mathbf{p}}(\mathbf{w}'')$ if and only if either both $(\mu, \mathbf{p}, \mathbf{w}') \in \Sigma^\infty$ and $(\mu, \mathbf{p}, \mathbf{w}'') \in \Sigma^\infty$ or none of them. Denote the partition over Ω that indicates the type of worker $\mu^{-1}(j)$ by $O^{\{\mu^{-1}(j)\}}$, i.e., $O^{\{\mu^{-1}(j)\}}(\mathbf{w}') = O^{\{\mu^{-1}(j)\}}(\mathbf{w}'')$ if and only if $\mathbf{w}'(\mu^{-1}(j)) = \mathbf{w}''(\mu^{-1}(j))$. Now we define the partition profile Π as follows:

$$\Pi_j(\mathbf{w}') := O^{\{\mu^{-1}(j)\}}(\mathbf{w}') \cap T_{\mu,\mathbf{p}}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{ and all } j \in J. \tag{14}$$

Since $(\mu, \mathbf{p}, \mathbf{w})$ is not Σ^∞-blocked, we know, by checking Definition 3 and 4, that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is not blocked. Similarly, $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$.

Consider a particular partition profile Π^0 defined as

$$\Pi^0_j(\mathbf{w}') := O^{\{\mu^{-1}(j)\}}(\mathbf{w}') \text{ for all } \mathbf{w}' \in \Omega \text{and all } j \in J.$$

Then $(\mu, \mathbf{p}, \mathbf{w}, \Pi^0)$ is not blocked by Fact 1 (otherwise $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is blocked since Π is finer than Π^0). Similarly, $(\mu, \mathbf{p}, \mathbf{w}', \Pi^0)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$.

Therefore, $N^{(\mu,\mathbf{p},\Pi^0)}(\mathbf{w}) \supset T_{\mu,\mathbf{p}}(\mathbf{w})$. As a result, $[H_{\mu,\mathbf{p}}(\Pi^0)]_j(\mathbf{w}') = \Pi^0_j(\mathbf{w}')$ for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$ and all $j \in J$. Since $(\mu, \mathbf{p}, \mathbf{w}', \Pi^0)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$, we know that $(\mu, \mathbf{p}, \mathbf{w}', H_{\mu,\mathbf{p}}(\Pi^0))$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$, which implies that $N^{(\mu,\mathbf{p},H_{\mu,\mathbf{p}}(\Pi^0))}(\mathbf{w}) \supset T_{\mu,\mathbf{p}}(\mathbf{w})$. Keep applying this argument and define $\Pi^k := H_{\mu,\mathbf{p}}(\Pi^{k-1})$ for $k = 1, 2, \dots,$ until we find a fixed point of $H_{\mu,\mathbf{p}}$. This must be done within finitely many times because the partition profile gets finer whenever it is not a fixed point. Denote the fixed point by Π^∞. Then induction shows that $N^{(\mu,\mathbf{p},\Pi^\infty)}(\mathbf{w}) \supset T_{\mu,\mathbf{p}}(\mathbf{w})$ and that $(\mu, \mathbf{p}, \mathbf{w}', \Pi^\infty)$ is not blocked for all $\mathbf{w}' \in T_{\mu,\mathbf{p}}(\mathbf{w})$, particularly for $\mathbf{w}' = \mathbf{w}$.

Sufficiency. Suppose there exists a partition profile Π such that $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ is stable. Let $\Sigma^N := \{(\mu, \mathbf{p}, \mathbf{w}') : \mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})\}$. Then obviously $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^N$. By definition of $N^{(\mu,\mathbf{p},\Pi)}$, $(\mu, \mathbf{p}, \mathbf{w}')$ is not Σ^N-blocked for all $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$. (Otherwise, $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ is blocked by Fact 1, contradicting $\mathbf{w}' \in N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w})$.) Therefore, Σ^N is a self-stabilizing set. By Proposition 2 of LMPS, $(\mu, \mathbf{p}, \mathbf{w}) \in \Sigma^N \subset \Sigma^\infty$.

C Efficiency of Stable States

To establish the efficiency of stable states under the assumptions of monotonicity and supermodularity, we will need the monetary unit, denoted by 1 that measures ϕ and ν, to be small enough to satisfy

$$
\begin{aligned}
& 0 < 1 < \frac{1}{5}\min\{D_1, D_2, D_3, D_4, D_5\}, \qquad (15)\\
\textit{where } & D_1 = \min_{w;f} \nu_{wf} + \phi_{wf} \ \textit{s.t.}\ \nu_{wf} + \phi_{wf} > 0,\\
& D_2 = \min_{w<w'} \nu_{w'f} - \nu_{wf},\\
& D_3 = \min_{f<f'} \phi_{wf'} - \phi_{wf},\\
& D_4 = \min_{w<w';f<f'} (\nu_{w'f'} - \nu_{w'f}) - (\nu_{wf'} - \nu_{wf}),\\
\textit{and } & D_5 = \min_{w<w';f<f'} [(\phi_{w'f'} + \nu_{w'f'}) - (\phi_{w'f} + \nu_{w'f})] - [(\phi_{wf'} + \nu_{wf'}) - (\phi_{wf} + \nu_{wf})].
\end{aligned}
$$

If $\nu_{wf} + \phi_{wf} \le 0$ for all $w \in W$ and $f \in F$, then no one has the incentive to be matched; otherwise D_1 is well defined and strictly positive because both W and F are finite. D_2 and D_3 are well defined and strictly positive because of Assumption 2, while D_4 and D_5 because of Assumption 3. We discuss only one efficiency result of LMPS while omit the similar but tedious analysis for others. The following Theorem 3′ says that every stable allocation is efficient if the monetary unit is small enough. The proof of Theorem 3 incorporates discreteness of payments into that of LMPS's Proposition 3.[18]

[18] Another difference from LMPS's proof is that we re-label workers and firms according to their types, both in descending order. This makes no essential change from LMPS's proof while shortens the presentation.

Theorem 3′. *Under Assumptions 1-3, every stable allocation is efficient if the monetary unit satisfies condition (15).*

Proof. Let $I = \{1, \dots, K\}$ and $J = \{1, \dots, L\}$. Re-label all workers and all firms such that

$$\mathbf{w}(K) \leq \mathbf{w}(K-1) \leq \cdots \leq \mathbf{w}(1) \text{ and}$$
$$\mathbf{f}(L) \leq \mathbf{f}(L-1) \leq \cdots \leq \mathbf{f}(1)$$

Without loss of generality, when more than one worker (firm) possess the same type, given a stable allocation $(\mu, \mathbf{p})$, re-label them according to their partner's type in descending order within the same-type group.

Step 1. For each $k \in I$, if $\mu(k) \neq \emptyset$, then $\mu(k') \neq \emptyset$ for all $k' < k$.

Suppose to the contrary that $\mu(k) \neq \emptyset$ while $\exists k' < k$ s.t. $\mu(k') = \emptyset$. By our re-labelling rule, $\mathbf{w}(k'+1) < \mathbf{w}(k')$. Let k' be the largest identity so that $\mu(k'+1) = l \neq \emptyset$ for some l. Denote the payment from l to $k'+1$ by p. We proceed to construct a $\tilde{p}$ such that (k', l) is a blocking pair at $\tilde{p}$. Take an integer $\tilde{p}$ such that $-1 - \nu_{\mathbf{w}(k'+1),\mathbf{f}(l)} \leq \tilde{p} < -\nu_{\mathbf{w}(k'+1),\mathbf{f}(l)}$. So for worker k',

$$\nu_{\mathbf{w}(k'),\mathbf{f}(l)} + \tilde{p} \geq \nu_{\mathbf{w}(k'),\mathbf{f}(l)} - \nu_{\mathbf{w}(k'+1),\mathbf{f}(l)} - 1 > 0$$

by Assumption 2 and condition (15), and for firm l,

$$\begin{aligned}\phi_{\mathbf{w}(k'),\mathbf{f}(l)} - \tilde{p} &> \phi_{\mathbf{w}(k'),\mathbf{f}(l)} + \nu_{\mathbf{w}(k'+1),\mathbf{f}(l)} \\ &\geq \phi_{\mathbf{w}(k'+1),\mathbf{f}(l)} + \nu_{\mathbf{w}(k'+1),\mathbf{f}(l)} \\ &\geq \phi_{\mathbf{w}(k'+1),\mathbf{f}(l)} - p\end{aligned}$$

since $\nu_{\mathbf{w}(k'+1),\mathbf{f}(l)} + p \geq 0$. The following condition,

$$\nu_{w,\mathbf{f}(l)} + \tilde{p} < \nu_{w,\mathbf{f}(l)} - \nu_{\mathbf{w}(k'+1),\mathbf{f}(l)} < 0 \text{ for all } w < \mathbf{w}(k'+1),$$

ensures that no worker of type lower than $\mathbf{w}(k'+1)$ would like to participate this blocking, which implies that firm l will better off without uncertainty. Therefore, (k', l) is a blocking pair at payment $\tilde{p}$, contradicting the stability of $(\mu, \mathbf{p})$.

Step 2. For each $l \in J$, if $\mu^{-1}(l) \neq \emptyset$, then $\mu^{-1}(l') \neq \emptyset$ for all $l' < l$.

Suppose to the contrary that $\mu^{-1}(l) \neq \emptyset$ while $\exists l' < l$ s.t. $\mu^{-1}(l') = \emptyset$. By our re-labelling rule, $\mathbf{f}(l'+1) < \mathbf{f}(l')$. Let l' be the largest identity so that $\mu^{-1}(l'+1) = k \neq \emptyset$ for some k. Denote the payment from $l'+1$ to k by p. We proceed to show that (k, l') is a blocking pair at $p+1$. So we have

$$\nu_{\mathbf{w}(k),\mathbf{f}(l')} + p + 1 > \nu_{\mathbf{w}(k),\mathbf{f}(l'+1)} + p.$$

Because firm l' can observe the stable allocation, particularly the individual rationality of $l'+1$, we have (by strictly increasing ϕ_{wf} in f and condition (15))

$$\phi_{\mathbf{w}(k),\mathbf{f}(l')} - p - 1 > \phi_{\mathbf{w}(k),\mathbf{f}(l'+1)} - p \geq 0,$$

which is independent of worker k's type. Therefore, (k, l') is a blocking pair at $p+1$, contradicting the stability of $(\mu, \mathbf{p})$.

Step 3. Obviously, there exists r such that $\mu(\{1, \dots, r\}) = \{1, \dots, r\}$.

Step 4. $\mu(r) = r$.

Suppose matches in the stable allocation are

$$\begin{array}{cccc} \textit{worker } t & \dots & \textit{worker } r & \dots \\ \underline{p} & \dots & \bar{p} & \dots \\ \textit{firm } r & \dots & \textit{firm } s & \dots \end{array}$$

satisfying that $\mathbf{w}(r) < \mathbf{w}(t)$ and $\mathbf{f}(r) < \mathbf{f}(s)$. We proceed to show that either (r, r) or (t, s) is a blocking pair.

For (r, r) being a blocking pair, we need an integer $\tilde{p}$ such that

$$\begin{aligned} \nu_{\mathbf{w}(r)\mathbf{f}(r)} + \tilde{p} &> \nu_{\mathbf{w}(r)\mathbf{f}(s)} + \bar{p} \textit{ and} \\ \phi_{\mathbf{w}(r)\mathbf{f}(r)} - \tilde{p} &> \phi_{\mathbf{w}(t)\mathbf{f}(r)} - \underline{p}, \\ \textit{i.e. } \nu_{\mathbf{w}(r)\mathbf{f}(s)} - \nu_{\mathbf{w}(r)\mathbf{f}(r)} + \bar{p} < \tilde{p} &< \phi_{\mathbf{w}(r)\mathbf{f}(r)} - \phi_{\mathbf{w}(t)\mathbf{f}(r)} + \underline{p}, \end{aligned}$$

which can be achieved if

$$\left(\nu_{\mathbf{w}(r)\mathbf{f}(s)} - \nu_{\mathbf{w}(r)\mathbf{f}(r)}\right) - \left(\phi_{\mathbf{w}(r)\mathbf{f}(r)} - \phi_{\mathbf{w}(t)\mathbf{f}(r)}\right) + 2 < \underline{p} - \bar{p}. \tag{16}$$

For (t, s) being a blocking pair, we need an integer $\tilde{p}$ such that

$$\begin{aligned} \nu_{\mathbf{w}(t)\mathbf{f}(s)} + \tilde{p} &> \nu_{\mathbf{w}(t)\mathbf{f}(r)} + \underline{p} \\ \phi_{\mathbf{w}(t)\mathbf{f}(s)} - \tilde{p} &> \phi_{\mathbf{w}(r)\mathbf{f}(s)} - \bar{p} \textit{ and} \\ \nu_{\mathbf{w}(t+1)\mathbf{f}(r)} + \underline{p} &> \nu_{\mathbf{w}(t+1)\mathbf{f}(s)} + \tilde{p}, \\ \textit{i.e. } \nu_{\mathbf{w}(t)\mathbf{f}(r)} - \nu_{\mathbf{w}(t)\mathbf{f}(s)} + \underline{p} < \tilde{p} &< \min\{\phi_{\mathbf{w}(t)\mathbf{f}(s)} - \phi_{\mathbf{w}(r)\mathbf{f}(s)} + \bar{p}, \\ &\qquad \nu_{\mathbf{w}(t+1)\mathbf{f}(r)} - \nu_{\mathbf{w}(t+1)\mathbf{f}(s)} + \underline{p}\}, \end{aligned} \tag{17}$$

which can be achieved if

$$\underline{p} - \bar{p} < \left(\phi_{\mathbf{w}(t)\mathbf{f}(s)} - \phi_{\mathbf{w}(r)\mathbf{f}(s)}\right) - \left(\nu_{\mathbf{w}(t)\mathbf{f}(s)} - \nu_{\mathbf{w}(t)\mathbf{f}(r)}\right) - 2. \tag{18}$$

Condition (17) rules out any worker t type that is lower than $\mathbf{w}(t)$ because of the strict supermodularity of ν_{wf}. We simplified the minimizing function because of again the supermodularity of ν_{wf} and condition (15).

Since by condition (15)

$$[(\phi_{\mathbf{w}(t),\mathbf{f}(s)} + \nu_{\mathbf{w}(t),\mathbf{f}(s)}) - (\phi_{\mathbf{w}(t),\mathbf{f}(r)} + \nu_{\mathbf{w}(t),\mathbf{f}(r)})] \\ -[(\phi_{\mathbf{w}(r),\mathbf{f}(s)} + \nu_{\mathbf{w}(r),\mathbf{f}(s)}) - (\phi_{\mathbf{w}(r),\mathbf{f}(r)} + \nu_{\mathbf{w}(r),\mathbf{f}(r)})] > 5,$$

at least one of (16) and (18) is satisfied, contradicting the stability of $(\mu, \mathbf{p})$.

Step 5. Inductively, $\mu(i) = i$ for all $i \in \{1, \ldots, r\}$, i.e., every stable allocation is positive assortative in type.

The rest of this proof is similar to that of (Liu et al., 2014, pp. 583), where condition (15) will be used again to show that (i) if $\nu_{wf} + \phi_{wf} > 0$, then $\nu_{wf} + \phi_{wf} > 5$; and that (ii) $\nu_{wf} - \max_{w'<w} \nu_{w'f} > 5$. □

References

Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, pages 729–747.

Balinski, M. and Sönmez, T. (1999). A tale of two mechanisms: student placement. *Journal of Economic Theory*, 84(1):73–94.

Bikhchandani, S. (2017). Stability with one-sided incomplete information. *Journal of Economic Theory*, 168:372–399.

Chakraborty, A., Citanna, A., and Ostrovsky, M. (2010). Two-sided matching with interdependent values. *Journal of Economic Theory*, 145(1):85–105.

Chen, B., Fujishige, S., and Yang, Z. (2010). Decentralized market processes to stable job matchings with competitive salaries. *KIER Discussion Paper*, 749.

Chen, B., Fujishige, S., and Yang, Z. (2016). Random decentralized market processes for stable job matchings with competitive salaries. *Journal of Economic Theory*, 165:25–36.

Chen, Y.-C. and Hu, G. (2017). A theory of matching with incomplete information. *Working Paper*.

Crawford, V. P. and Knoer, E. M. (1981). Job matching with heterogeneous firms and workers. *Econometrica*, pages 437–450.

Ehlers, L. and Massó, J. (2007). Incomplete information and singleton cores in matching markets. *Journal of Economic Theory*, 136(1):587–600.

Fujishige, S. and Yang, Z. (2016). Decentralised random competitive dynamic market processes. *University of York*, Discussion Papers in Economics.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

Kelso, A. S. J. and Crawford, V. P. (1982). Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society*, pages 1483–1504.

Klaus, B. and Klijn, F. (2007a). Corrigendum to "on randomized matching mechanisms"[economic theory 8 (1996) 377–381]. *Economic Theory*, 32(2):411–416.

Klaus, B. and Klijn, F. (2007b). Paths to stability for matching markets with couples. *Games and Economic Behavior*, 58(1):154–171.

Knuth, D. E. (1976). *Mariages stables et leurs relations avec d&autres problèmes combinatoires*. Presses de l'Université de Montréal.

Kojima, F. and Ünver, M. U. (2008). Random paths to pairwise stability in many-to-many matching problems: a study on market equilibration. *International Journal of Game Theory*, 36(3-4):473–488.

Lauermann, S. and Nöldeke, G. (2014). Stable marriages and search frictions. *Journal of Economic Theory*, 151:163–195.

Lazarova, E. and Dimitrov, D. (2017). Paths to stability in two-sided matching under uncertainty. *International Journal of Game Theory*, 46(1):29–49.

Liu, Q., Mailath, G. J., Postlewaite, A., and Samuelson, L. (2014). Stable matching with incomplete information. *Econometrica*, 82(2):541–587.

Ma, J. (1996). On randomized matching mechanisms. *Economic Theory*, 8(2):377–381.

Mailath, G. J., Postlewaite, A., and Samuelson, L. (2013). Pricing and investments in matching markets. *Theoretical Economics*, 8:535–590.

Mailath, G. J., Postlewaite, A., and Samuelson, L. (2017). Premuneration values and investments in matching markets. *The Economic Journal*.

Pomatto, L. (2015). Stable matching under forward-induction reasoning. *Working Paper*.

Roth, A. E. (1989). Two-sided matching with incomplete information about others' preferences. *Games and Economic Behavior*, 1(2):191–209.

Roth, A. E. (2008). Deferred acceptance algorithms: History, theory, practice, and open questions. *International Journal of Game Theory*, 36(3-4):537–569.

Roth, A. E. and Sotomayor, M. A. O. (1990). *Two-sided matching: A study in game-theoretic modeling and analysis*. Number 18. Cambridge University Press.

Roth, A. E. and Vande Vate, J. H. (1990). Random paths to stability in two-sided matching. *Econometrica*, pages 1475–1480.

Shapley, L. S. and Shubik, M. (1971). The assignment game i: The core. *International Journal of Game Theory*, 1(1):111–130.