A Theory of Stability in Matching with Incomplete Information

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Abstract

We provide a framework for studying two-sided matching markets with incomplete information. The framework accommodates two-sided incomplete information as well as heterogeneous information among the agents. We propose a notion called stability for a market state, which, based upon agents’ information structure, requires (i) individual rationality, (ii) no blocking and (iii) information stability. The novelty of our stability notion lies in how the agents evaluate a blocking prospect, in the presence of general two-sided incomplete information. We show that a stable state exists; moreover, if a state is stable, then coarsening agents’ information leads to another stable state.

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1 Introduction

This paper provides a framework for studying two-sided matching markets with incomplete information. Matching is one of the important functions of markets (Roth (2008)). Since the seminal work of Gale and Shapley (1962) and Shapley and Shubik (1971), for several decades, stability has been a key notion used in both the theoretical study and the practical design of matching markets. Conceptually, stability has been connected to both equity and efficiency, two of the most important notions in economics.\footnote{See, e.g., Abdulkadiroglu and Sönmez (2013) for how stability implies the elimination of justified envy, a basic fairness axiom; see, e.g., Shapley and Shubik (1971) for how stability leads to efficiency.} Practically, the design of many matching markets aims to achieve stable matchings.\footnote{See, e.g., Roth and Peranson (1999) and Abdulkadiroglu and Sönmez (2013) for more market design details.}

A prevailing assumption in this literature is that the information is complete, i.e., the characteristics and preferences of all market participants are publicly known. The assumption makes the analysis tractable but is at best an idealization. A woman or man may have imprecise information about her or his potential partner in a marriage market. A college may have imprecise information about the students’ capability and a student may also have imperfect understanding about her/his suitability to a college. Likewise, a firm may not know the productivity of its job candidates, whereas a worker may not know the corporate culture or financial health of his potential employer.

Incomplete information is also important concerning the theory of market design. In particular, there is no mechanism that yields stable matchings and ensures that agents on both sides of the market have dominant-strategy incentive to truthfully reveal their information (Roth (1982)). For example, the man-proposing deferred acceptance mechanism makes it a dominant strategy for men to truthfully reveal their information, but not the women; and vice versa. Even if the notion of incentive compatibility or the notion of stability is weakened to
help getting existence, stable mechanisms can only exist under stringent assumptions; see, e.g., Ehlers and Massó (2007) and Yenmez (2013). Hence, incomplete information, especially incomplete information on both sides of the market, is of substantive importance.

Despite the importance, the literature is yet to provide a general analytical framework for stable matching with incomplete information.\(^3\) In this paper, we propose such a framework along with a new notion of stability. Equipped with the framework, we establish the existence of a stable matching and study the properties and the structure of stable matchings. Our analytical framework is applicable to a broad range of matching setups with incomplete information.

To motivate our framework, consider a setup where workers are matched with firms.\(^4\) We first propose a solution concept called (incomplete-information) stability. Specifically, stability under complete information requires individual rationality (i.e., no one strictly prefers staying unmatched to the status quo) and no blocking pair (i.e., no worker and firm would both prefer being matched with each other at some wage to the status quo). When a firm has incomplete information about workers’ types, however, she would not know whether hiring another worker is better than keeping her current employee. Moreover, the firm may not even know whether keeping her current employee is better than becoming

\(^3\)See Section 1.1 for a discussion about related papers such as Liu et al. (2014), Bikhchandani (2017), Chen and Hu (2020), Pomatto (2019) and Liu (2020). All of them assume that the agents have complete information about one side of the market. Namely, the types of agents on one side (say, the firms) are common knowledge. Two notable exceptions are Bikhchandani (2014) and Liu (2017). In defining a blocking pair when there are two-sided incomplete information, both of them incorporated the idea of reciprocal consideration between the worker and the firm. Specifically, the blocking notion of Bikhchandani (2014) imposed assumptions on both the type space (e.g., independence) and the value functions (e.g., monotonicity) so as to define the two agents’ most optimistic assumptions about each other’s type. Blocking of Liu (2017) is defined as a self-fulfilling notion, i.e., the two agents’ willingness to form a new partnership is based on beliefs that are confirmed by their willingness. Unlike these two papers, we formulate agents’ information through partitions over type profiles and describe agents’ willingness according to consideration refinements. See Section 3.2.

\(^4\)We adopt this job-market setting from the recent papers on matching with incomplete information (e.g., Liu et al. (2014) and Liu (2020)). It will become clear that our notions and analysis are also applicable to other matching setups such as those without transferable utility and/or with ordinal preferences.
unmatched. Similar difficulties arise on the worker side. In this situation, the existing notions of individual rationality, blocking, and stability in the complete-information environment become inadequate.

In the incomplete-information environment, each agent is associated with a type. The types of the agents determine their payoffs from a match. Moreover, an agent’s information is described by a partition over the possible type profiles of all agents. In our setting, a state of the market consists of an allocation (i.e., a matching and wage profile) and an information structure (i.e., a true type profile and a partition profile). A state is stable if (i) the allocation is individually rational with respect to the information structure, (ii) the allocation admits no blocking pair with respect to the information structure, and (iii) individual rationality and the absence of blocking convey no further information to the agents. We establish the existence of stable states; see Theorem 1.

The novelty of our stability notion lies in how the agents evaluate a blocking prospect in requirement (ii). When the firms’ types are common knowledge as in Liu et al. (2014), a firm’s evaluation of a new partnership with a worker is conditional on the worker’s willingness of joining it. However, the worker’s evaluation does not depend on the firm’s evaluation, since the worker knows the type of the firm and thereby his potential payoff.

When incomplete information prevails on both sides, each agent in a potential blocking pair needs to consider not only her/his potential partner’s “willingness” but also how her/his potential partner considers her/his “willingness”, and so on. Unlike the case with one-sided incomplete information, it is not a priori clear how “willingness” should be defined to start with. Our key observation is that a firm can safely rule out a worker’s type from consideration if in no circumstance will the worker’s type be willing to partner with the firm. The consideration of the worker would never restore the worker type’s willingness if the type concludes no benefit even without factoring in the firm’s willingness to narrow down the circumstances/firm types in consideration. Once the pair of
agents rule out such types from each other, they will consider fewer possible types of each other and there could be more types who in no (now fewer) circumstances will benefit from the blocking and thereby be ruled out by the other side, and so on. We call the limit set of type profiles a consideration set of a blocking agent. Like Liu et al. (2014), we require that each of the two agent benefit from the blocking prospect under every type profile in her/his consideration set.

This refinement of consideration set is consistent with and generalizes the belief-free formulation in Liu et al. (2014) to matchings with two-sided incomplete information. More precisely, in Liu et al. (2014), a matching outcome is deemed unstable in their iteration procedure if and only if it shall be blocked for any possible belief of the firm conditional on the willingness of the worker. Similarly, a type profile is ruled out from a firm’s consideration if and only if the worker never prefers the new partnership for any belief consistent with the worker’s consideration.\(^5\)

Despite the similar belief-free formulation, we ought to point out that the iteration process of defining a consideration set is crucially different from the iteration process which Liu et al. (2014) invoke in defining the set of stable outcomes. To wit, the latter iteration process is to ensure the information stability requirement in (iii) that individual rationality and absence of blocking should convey no further information to the agents. Clearly, the definition of consideration set, which is involved in defining the blocking notion, precedes and determines what the agents learn from the absence of blocking. In particular, a firm in Liu et al. (2014), thanks to one-sided incomplete information, can obtain her consideration set upon conditioning on her potential partner’s willingness without appealing to further iteration.

Our notions of blocking and stability lend themselves to clear epistemic foundations. Namely, a state is blocked by a worker and a firm if and only if each of them knows according the consideration set that s/he can benefit from

\(^5\)See Sections 1.1 and 3.3 of Liu et al. (2014) for more discussions. See also Bergemann and Morris (2007) for belief free solution concepts in games with incomplete information.
the new partnership. In a similar vein, a state is stable if and only if it is common knowledge that the state is individually rational and not blocked. (See Propositions 1-2.)

We emphasize two properties of blocking, both of which become vacuous when information is complete. First, improvement at the truth (IT) says that agents in the blocking coalition will obtain higher \textit{ex post} payoffs. Obviously, if IT is satisfied, then blocking implies complete-information blocking. IT bridges the connection between the incomplete-information stability and the complete-information stability. Second, information monotonicity (IM) says that blocking opportunities are easier to arise when agents have more precise information. Our blocking notion satisfies both properties; see Theorem 2.

We investigate the structure of stable states and show that if a state is stable, then coarsening agents’ information leads to another stable state. (See Proposition 3.) The result implies that any stable matching outcome (i.e., the allocation and the type profile in a stable state) can be supported as a stable state by a specific partition profile—a partition profile induced by the trivial partition under which agents have the least information; see Theorem 3.

Finally, we discuss two alternative blocking/stability notions. The first notion, called naive blocking, represents the situation where the agents do not refine their consideration set, or equivalently, their consideration set contains all types profiles which they cannot rule out under their information partitions. Naive blocking represents the most conservative blocking notion under incomplete information. We identify conditions under which naive blocking is equivalent to blocking (Theorem 4). In contrast, the second notion, called sophisticated blocking, represents the situation where agents are more aggressive in refining their consideration set than they are under blocking. Specifically, we argue that there are type profiles which the agents can ruled out under some additional assumptions on consideration set refinement. We also examine the properties IT and IM for the two alternative blocking notions. Contrary to the relationship
among the notions of blocking, each sophisticatedly stable state is a stable state, and each stable state is a naively stable state. Some of these properties will be formally stated and proved only in the online appendix, where we also show that under the conditions identified in Theorem 4, naive blocking and blocking are equivalent to sophisticated blocking.

The rest of the section reviews the related literature. Section 2 introduces the model. Section 3 defines individual rationality, blocking and stability with incomplete information, and also documents the epistemic foundations of those definitions. Section 4 examines improvement at the truth and information monotonicity. Section 5 studies alternative blocking/stability notions. Section 6 concludes with future research directions. All proofs can be found in the appendix. More discussions on alternative blocking/stability notions can be found in the online appendix.

1.1 Related literature

Our paper is related to three strands of literature, the classical matching theory, the recent development on matching with incomplete information, and the literature on core with incomplete information. First, the seminal paper by Gale and Shapley (1962), studying the marriage matching market and the college admission market, has led to a sizable literature on two-sided matching; see Roth and Sotomayor (1990) for a survey on classical theories and Abdulkadiroglu and Sönmez (2013) and Kojima (2015) on recent developments. Our framework for matching with incomplete information coincides with the classical models when the agents have complete information.

Second, Liu et al. (2014) studied stable matchings with one-sided incomplete information. They proposed a stability notion and showed that under certain monotonicity and supermodularity assumptions on the agents’ prenumeration values from a match, every stable matching outcome is efficient. Pomatto (2019) considers a noncooperative matching game and uses forward-induction reasoning to derive the set of stable outcomes that is identified in Liu et al. (2014).
Bikhchandani (2017) proposes a notion of stability which is similar to that of Liu et al. (2014) but applies to a Bayesian setting with nontransferable utilities. Bikhchandani (2017) points out that a Bayesian stable matching does not necessarily exist.\(^6\) Liu (2020) proposes a stability criterion that requires the Bayesian consistency of three beliefs; namely, the exogenously given prior beliefs, the off-path beliefs conditional on counterfactual pairwise blockings, and the on-path beliefs for stable matchings in the absence of such blockings.\(^7\)

All these papers assume one-sided incomplete information. In this aspect, our paper contributes to this literature by modeling matching with two-sided incomplete information. Our main novelty lies in the formulation of consideration sets in defining the blocking notion. The formulation involves an iterative reasoning, which becomes trivial when there is only one-sided incomplete information. Our epistemic characterization of the stability notion is based on the framework of Aumann (1976). We present the characterizations of stability as well as the new notion of blocking in the context of two-sided incomplete information.

Finally, our blocking/stability notion is related to the literature on core with incomplete information. Specifically, Wilson (1978) proposed two notions of core which correspond to two polar cases of information-sharing rules within a blocking coalition. In one extreme case, the agents in a coalition pool all the information which they possess, and in the other extreme case, the agents in a coalition share no information with others. Subsequent papers such as Volij (2000) and Dutta and Vohra (2005) have explored different information sharing rules between the two polar cases. In particular, Volij (2000) formulates a blocking notion in which

\(^6\)See also Alston (2020).

\(^7\) Another strand of literature mainly focuses on stable mechanisms (instead of stable matchings) which also involve incomplete information. See, e.g., Roth (1989), Chakraborty, Citanna and Ostrovsky (2010), Yenmez (2013) and Ehlers and Massó (2007, 2015). A stable mechanism usually takes a notion of stable matchings as given and specifies information eliciting and matching producing rules to achieve stable matchings. From another angle, stable mechanisms are associated with centralized markets, while our paper aims to understand stability of matchings regardless of whether a centralized mechanism is available or not; see, e.g., Chen and Hu (2020) for a formulation of a decentralized “path-to-stability” problem with incomplete information.
agents in a coalition iteratively communicate their willingness to join a blocking until all of them reach a consensus of improvement with respect to their updated information from the communication. In contrast, our notions of blocking and stability rely on an iterative reasoning in which information updating is only hypothetical; see Section 3.2.2 for more discussion.

2 The Model

We consider the following setup of matching with incomplete information, which extends the ones of Liu et al. (2014) and Chen and Hu (2020).

There is a finite set $I$ of workers to be matched with a finite set $J$ of firms. Denote a generic worker by $i$ and a generic firm by $j$. Also denote a generic agent by $k$ when we do not distinguish workers from firms. While each agent’s index $i$ or $j$ is publicly observed, the agent’s productivity is determined by the agent’s private type. Let $W$ be the finite set of worker types and $F$ be the finite set of firm types. A type assignment for firms is a mapping $f : J \rightarrow F$, and similarly a type assignment for workers is another mapping $w : I \rightarrow W$. We denote by $t := (w, f)$ a generic type assignment, and denote by $T$ a set of type assignments, i.e., $T \subset W^{\mid I \mid} \times F^{\mid J \mid}$.

A match between a worker of type $w \in W$ and a firm of type $f \in F$ gives rise to the worker premuneration value $\nu_{w,f} \in \mathbb{R}$ and the firm premuneration value $\phi_{w,f} \in \mathbb{R}$. The sum of $\nu_{w,f}$ and $\phi_{w,f}$ is called the surplus of the match. The functions $\nu : W \times F \rightarrow \mathbb{R}$ and $\phi : W \times F \rightarrow \mathbb{R}$ are common knowledge among the agents. Denote these values by $\nu_{t(i), t(j)}$ for unmatched worker $i$ and $\phi_{t(j), t(j)}$ for unmatched firm $j$, both of which are set to be zero. Given a match between worker $i$ and firm $j$, with type assignment $t$, the worker’s payoff and the firm’s payoff under wage $p \in \mathbb{R}$ are, respectively, $\nu_{t(i), t(j)} + p$ and $\phi_{t(i), t(j)} - p$.

---

8See Mailath, Postlewaite and Samuelson (2013, 2017) for discussions on premuneration values.
A matching is a mapping $\mu : I \cup J \to I \cup J$ such that (i) $\mu(i) \in J \cup \{i\}$; (ii) $\mu(j) \in I \cup \{j\}$; and (iii) $\mu(i) = j$ if and only if $\mu(j) = i$ for all $i \in I$ and all $j \in J$. In words, each worker is either unmatched, denoted by $\mu(i) = i$, or assigned to a firm that employs him; and each firm is either unmatched, denoted by $\mu(j) = j$, or hires a worker. A payment scheme $\mathbf{p}$ associated with a matching $\mu$ is a vector that specifies a wage payment $p_{i,\mu(i)} \in \mathbb{R}$ to each worker $i$, and a wage payment $p_{\mu(j),j} \in \mathbb{R}$ from each firm $j$. To avoid nuisance cases, we associate zero payments with unmatched agents, by setting $p_{ii} = p_{jj} = 0$. Finally, an allocation $(\mu, \mathbf{p})$ consists of a matching $\mu$ and an associated payment scheme $\mathbf{p}$. We assume that the allocation is publicly observable.

Beyond the public information that the agents’ type assignment belongs to $T$, the agents may also have their own private information about the type assignment. Specifically, for every agent $k$, we describe her private information by a partition $\Pi_k$ over $T$. For any type assignment $t$, write $\Pi_k(t)$ as the cell of partition $\Pi_k$ that contains $t$. When the true type assignment is $t$, agent $k$ regards each type assignment $t'$ in $\Pi_k(t)$ as possible. Denote the profile of partitions by $\Pi$, i.e., $\Pi := (\{\Pi_i\}_{i \in I}, \{\Pi_j\}_{j \in J})$, which is assumed to be common knowledge. Say partition profile $\Pi'$ is (weakly) finer than partition profile $\Pi$ if, for each agent $k$, we have $\Pi'_k(t) \subset \Pi_k(t)$ for every type assignment $t \in T$.

A state of the matching market, $(\mu, \mathbf{p}, t, \Pi)$ specifies an allocation $(\mu, \mathbf{p})$, a type assignment $t$, and a partition profile $\Pi$. In words, a state of the market describes who is matched with whom at which wage, and what each agent knows.

Finally, we introduce the following standard notions from game theory (see, e.g., Osborne and Rubinstein (1994)). An event is a subset of $T$. Say agent $k$ knows an event $E \subset T$ at $t$ if $\Pi_k(t) \subset E$. Given a partition profile $\Pi$, each agent $k$ has a knowledge function $\mathcal{K}_{k,\Pi}(\cdot)$ defined by

$$
\mathcal{K}_{k,\Pi}(E) := \{t \in T : \Pi_k(t) \subset E\}.
$$

An event $E' \subset T$ is self-evident among a set of agents $K \subset I \cup J$ if for all $t \in E'$ we have $\Pi_k(t) \subset E'$ for all $k \in K$. An event $E$ is said to be common knowledge
among a set of agents $K \subset I \cup J$ at $t$ if there is a self-evident event $E'$ among $K$ for which $t \in E' \subset E$.

We add a few remarks on the framework as a model for matching with incomplete information. First, the model apparently covers the classical complete information setup. Say information is complete if every agent knows what is the true type assignment, i.e., $\Pi_k (t) = \{ t \}$ for all $t \in T$ and all $k$. Second, the model also covers the setup of one-sided incomplete information analyzed in Liu et al. (2014), Chen and Hu (2020) and Pomatto (2019). Formally, these papers analyze the situation where the firms’ types are common knowledge, i.e., $f' = f$ for all $(w, f), (w', f') \in T$.

Third, unlike Liu et al. (2014) and Chen and Hu (2020), the model accommodates but makes no presumption about the agents’ knowledge about their partners’ types. For instance, if we assume that agents know their partner’s type, then for each state $(\mu, p, t, \Pi)$ with $\mu (i) = j$ and $t = (w, f)$, we have $w' (i) = w (i)$ for all $(w', f') \in \Pi_j ((w, f))$ and $f' (j) = f (j)$ for all $(w', f') \in \Pi_i ((w, f))$. Finally, the model also lends itself to a transparent epistemic description for the blocking/stability notion which we will demonstrate in the next section.

3 Stability: a leading notion
3.1 Individual rationality
A state is individually rational if for every agent, it is possible, based on the agent’s information partition, to have a nonnegative payoff.

**Definition 1** A state $(\mu, p, t, \Pi)$ is individually rational if for each $i \in I$ and each $j \in J$, respectively,

$$\nu_{\mu(i)} (v_{t'}) + p_{\mu(i)} \geq 0 \text{ for some } t' \in \Pi_i (t) \text{ and}$$

$$\phi_{\mu(j)} (v_{t'}) - p_{\mu(j), j} \geq 0 \text{ for some } t' \in \Pi_j (t).$$

Individual rationality has a natural epistemic foundation:

**Fact 1** A state $(\mu, p, t, \Pi)$ is individually rational if and only if at $t$, no agent
knows the event that s/he has a negative payoff, i.e., for all \( i \in I \) and all \( j \in J \),
\[
\{ t' \in T : \nu_{i}(t',\nu_{(i)}) + p_{i,\mu_{(i)}} < 0 \} \}
\] and
\[
\{ t' \in T : \phi_{j}(\nu_{(j)},\nu_{(j)}) - p_{\mu_{(j)},j} < 0 \} \}
\]

This is consistent with the perspective of Liu et al. (2014) according to which a blocking opportunity for a pair of agents must guarantee improvement under every type assignment which the two agents consider possible. We will also follow this perspective in defining our notion of blocking subsequently.

We hasten to discuss two special cases. First, if \( \Pi_{k}(t) = \{ t \} \) for every \( k \), then (1) and (2) reduce to the standard conditions for complete-information individual rationality. Second, Liu et al. (2014) assume that the two agents within a matched pair observe the true type of each other. In this case, (1) and (2) also reduce to the complete-information individual rationality.

3.2 Blocking

3.2.1 Motivation of consideration

With complete information, a matching is blocked if there exist a worker and a firm such that both agents can benefit from being rematched with each other at some wage. To motivate our notion of blocking under incomplete information, consider a potential rematching \((i, j; p)\) for the state \((\mu, p, t, \Pi)\). Observe that in evaluating the rematching with worker \( i \) at wage \( p \), firm \( j \) will not consider the type assignments in \( \Pi_{j}(t) \) at which worker \( i \) must not be willing to participate in the rematching with firm \( j \). That is, firm \( j \) can rule out from her consideration any type assignment \( t' \) at which worker \( i \) knows that he will not benefit from being rematched with firm \( j \) with wage \( p \). This is the key idea to define blocking when there is one-sided (worker-side) incomplete information as in Liu et al. (2014) and Chen and Hu (2020).

When the firm-side information is also incomplete, similarly, worker \( i \) will not consider type assignments in \( \Pi_{i}(t) \) at which firm \( j \) must not be willing to participate in the rematching with worker \( i \). Furthermore, with two-sided
incomplete information, once the pair of agents take into account each other’s ruling out some type assignments from their consideration, they may be able to rule out some more type assignments from their consideration, as we illustrate in the following example.

**Example 1** Consider a market with two workers, i.e., \( I = \{\alpha, \beta\} \), and two firms, i.e., \( J = \{a, b\} \). The set of possible type assignments is given by \( T = \{t, t', t''\} \), i.e.,

\[
\begin{array}{c|ccc}
 & t & t' & t'' \\
\hline
\alpha & 2 & 2 & 2 \\
\beta & 3 & 1 & 1 \\
a & 3 & 3 & 5 \\
b & 2 & 4 & 4 \\
\end{array}
\]

The premuneration values for workers and firms are given by the product form, i.e., \( \nu_{w,f} = \phi_{w,f} = w f \). Obviously, given any wage, every agent prefers a partner with a higher type to a partner with a lower type.

We first describe a state \((\mu, p, t, \Pi)\). Suppose that firm \(a\) hires worker \(\alpha\) and firm \(b\) hires worker \(\beta\). In other words, a matching \(\mu\) is given by \(\mu(\alpha) = a\) and \(\mu(\beta) = b\). Assume for simplicity that \(p = 0\). Suppose further that that the first type assignment \(t\) is true, and that agents’ partitions are given as if agents can observe their own types and their own partner’s types, i.e.,

\[
\begin{align*}
\Pi_\alpha &= \{\{t, t'\}, \{t''\}\}, \\
\Pi_\beta &= \{\{t\}, \{t', t''\}\}, \\
\Pi_a &= \{\{t, t'\}, \{t''\}\}, \text{ and} \\
\Pi_b &= \{\{t\}, \{t', t''\}\}.
\end{align*}
\]

It is straightforward to verify that if there is complete information at \(t\), then \(\beta\) and \(a\) will both benefit from being rematched with each other at wage 0. With incomplete information associated \(\Pi\), however, firm \(a\) would be worried about the possible type assignment \(t'\). To be precise, for the possible type 1 of
\( \beta \), firm \( a \) would obtain a lower payoff 3 than her status quo payoff 6.

However, we proceed to argue that the potential rematching \((\beta, a; 0)\) should block the state \((\mu, p, t, \Pi)\). First of all, worker \( \beta \) is willing to participate in the rematching \((\beta, a; 0)\) because he knows that the true type assignment is \( t \), at which he will get a higher payoff if rematched with firm \( a \). Then it suffices to check firm \( a \)'s willingness. Firm \( a \) only worries about the possible type assignment \( t' \). Nevertheless, we argue that she does not need to consider \( t' \). More precisely, she knows that if the true type assignment is \( t' \), then worker \( \beta \) regards both \( t' \) and \( t'' \) as possible. Note that worker \( \beta \) would not consider \( t'' \) if the true type assignment were \( t' \), because firm \( a \) would definitely object to the rematching \((\beta, a; 0)\) if \( t'' \) were the true type profile (i.e., there is no possibility for firm \( a \) to obtain a higher payoff). Hence, worker \( \beta \) would consider only \( t' \), at which he would object to the rematching \((\beta, a; 0)\). That is, worker \( \beta \) would definitely object to the rematching \((\beta, a; 0)\) should \( t' \) be the true type profile (i.e., there is no possibility under consideration for worker \( \beta \) to obtain a higher payoff). Therefore, firm \( a \) does not need to consider \( t' \). Considering only \( t \), firm \( a \) is willing to participate in the rematching \((\beta, a; 0)\). Therefore, \((\beta, a; 0)\) should block the state \((\mu, p, t, \Pi)\).

### 3.2.2 Consideration and blocking

To formalize the interactive reasoning in Example 1, we introduce a notion called consideration correspondences. To simplify notations, we fix a state \((\mu, p, t, \Pi)\) and a potential rematching \((i, j; p)\), and hereafter we omit the reference of the blocking notion to \((\mu, p, t, \Pi)\) and \((i, j; p)\). For each \( t' \in T \), we define the indicators \( \chi(t') \) for both agents as follows.

\[
\chi_i(t') := \begin{cases} 
Y & \text{if } \nu_{t'(i), t'(j)} + p > \nu_{t'(i), \mu(i)} + p_{i, \mu(i)}, \\
N & \text{otherwise}; 
\end{cases}
\]

\( (3) \)

\[
\chi_j(t') := \begin{cases} 
Y & \text{if } \phi_{t'(i), t'(j)} - p > \phi_{t'(j), \mu(j)} - p_{\mu(j), j}, \\
N & \text{otherwise}. 
\end{cases}
\]

\( (4) \)
For example, $\chi_i(t') = Y$ (Yes) means that worker $i$ is willing to participate in the blocking $(i,j;p)$ if he knows that the true type profile is $t'$. However, since worker $i$ may not know whether $t'$ is true, it should be clear that such worker $i$'s willingness is only hypothetical. This kind of hypothetical willingness/unwillingness constitutes the basis of our consideration refinement. We now define the notion of consideration correspondence as follows.

**Definition 2** Set $C^0 = \Pi$. For every $l \geq 1$ and every $t' \in T$, define the round-$l$ consideration correspondences

\[
C^l_i(t') := \{ t'' \in \Pi_i(t') : \exists t''' \in C^{l-1}_j(t'') \text{ s.t. } \chi_j(t''') = Y \} \quad \text{and} \quad (5)
\]

\[
C^l_j(t') := \{ t'' \in \Pi_j(t') : \exists t''' \in C^{l-1}_i(t'') \text{ s.t. } \chi_i(t''') = Y \} . \quad (6)
\]

The (limit) consideration correspondences for the potential rematching $(i,j;p)$ at $(\mu,p,t,\Pi)$ are defined as $C^\infty_i(t') = \cap_{l=1}^{\infty} C^l_i(t')$ and $C^\infty_j(t') = \cap_{l=1}^{\infty} C^l_j(t')$ for each $t' \in T$. We also call $C^\infty_k(t')$ the consideration set of agent $k$ at $t'$.

Two remarks on the consideration correspondences are in order. First, (5) and (6) mean that the worker $i$ and firm $j$ in the potential rematching only consider type assignments such that their potential partner “may be” willing to join the rematching, i.e., type assignments at which there is at least some “Yes” in their potential partner’s consideration set from the previous round. Equivalently, agent $k$ rules out a type assignment from her consideration in round $l$ if and only if agent $-k$ have confirmed his unwillingness at the type assignment in round $l - 1$. The confirmation of unwillingness means that agent $-k$’s consideration set in round $l - 1$ either consists of uniformly “No” or has become empty.

Here, the driving force for an agent to refine her consideration is the uniform $N$ indicators in some cell of her potential partner’s partition. To see this, let $C^l_k(\cdot)$ be a general consideration correspondence, not necessarily the one in Definition 2. A natural restriction on $C^l_k(\cdot)$ would be the following one, which says that consideration should be consistent with knowledge:

\[
C^l_k(t) \subset \Pi_k(t) \text{ for } k = i,j \text{ and all } t \in T. \quad (7)
\]

Although simple and seemingly redundant, this is the intrinsic assumption to
justify the consideration refinement process (5)-(6). More precisely, say, suppose
\(\chi_j(t') = N\) for all \(t' \in \Pi_j(t)\). Then whenever (7) is satisfied, we know that \(C_j^l(t')\) is either empty or consists exclusively of type assignments with \(N\) from firm \(j\). In either case, firm \(j\) will not join the rematching under \(t'\). Hence, worker \(i\) can safely ignore \(t'\).

Second, the following lemma verifies that the consideration correspondences are monotonically decreasing in \(l\). The monotonicity has two implications. First, it does not matter whether we require \(t'' \in \Pi_k(t')\) or \(t'' \in C_{k}^{l-1}(t')\) in (5) and (6). Second, since \(T\) is finite, there is some \(l^*\) such that \(C_i^l(t') = C_i^\infty(t')\) and \(C_j^l(t') = C_j^\infty(t')\) for all \(l \geq l^*\) and all \(t' \in T\). Note that \(C_k^\infty(t')\) might be empty.

**Lemma 1** For \(k = i, j\) and each \(t' \in T\), \(C_k^l(t')\) is decreasing in \(l\) w.r.t. set inclusion.

**Example 1 (Continuing)** Incorporating the information conveyed by indicators, we can rewrite the partitions of worker \(\beta\) and firm \(a\) as partitions over lists of indicators, i.e.,

\[
\begin{align*}
\Pi_\beta &= \{\{Y\}, \{N, Y\}\} \\
\Pi_a &= \{\{Y\}, \{N\}\}.
\end{align*}
\]

Worker \(\beta\)'s and firm \(a\)'s round-1 consideration correspondences are, respectively,

\[
\begin{align*}
C_\beta^1 : & \{t\} \quad \{t'\} \quad \{t''\} \\
C_a^1 : & \{t, t'\} \quad \{t', t''\}
\end{align*}
\]

We can represent such correspondences by deleting \(Y\) or \(N\) from the partitions over the lists of indicators, which facilitates the comparison with partitions:

\[
\begin{align*}
C_\beta^1 &= \{\{Y\}, \{N\}\} \\
C_a^1 &= \{\{Y\}, \{N\}\}.
\end{align*}
\]

\(^9\)In Section 5.2, we impose an additional assumption on consideration set refinement, with which, together with (7), agents can refine their consideration more aggressively than in (5)-(6).
In similar ways, worker $\beta$’s and firm $a$’s round-2 consideration correspondences can be written as follows, respectively:

\[
\begin{align*}
C^2_{\beta} &= \{\{Y\}, \{N\}\} \\
C^2_{a} &= \{\{Y\}\}, \{\}\}.
\end{align*}
\]

We say a state is blocked by a combination $(i, j; p)$ if both agents in it could have higher rematching payoffs under every type assignment that is considered at the true one.

**Definition 3** A state $(\mu, p, t, \Pi)$ is **blocked** by $(i, j; p)$ if $C^\infty_i(t) \neq \emptyset$, $C^\infty_j(t) \neq \emptyset$ and

\[
\begin{align*}
\nu_{t'}(i, j) + p &> \nu_{\mu(i)}(i, j) + p_{i, \mu(i)} \quad \text{for all } t' \in C^\infty_i(t) \text{ and} \quad (8) \\
\phi_{t'}(i, j) - p &> \phi_{\mu(j)}(i, j) - p_{i} \quad \text{for all } t' \in C^\infty_j(t). \quad (9)
\end{align*}
\]

In this case, we say $(i, j; p)$ is a **blocking combination** for $(\mu, p, t, \Pi)$. Say a state is **unblocked** if it is not blocked by any combination $(i, j; p)$.

As a special case, if $C^\infty_i(t) = \{t\}$ and $C^\infty_j(t) = \{t\}$, then (8)-(9) reduce to the standard conditions for complete-information blocking. Definition 3 also extends the definitions of blocking in Liu et al. (2014) and Chen and Hu (2020) to two-sided incomplete information while remaining “belief-free” in two important aspects. First, a type assignment is ruled out from the agents’ consideration set if and only if the potential partner entertains no possibility of joining the blocking combination, i.e., the potential partner’s consideration is either empty or consists exclusively of “No”. Second, the agents form a blocking combination if and only if they secure strict improvement in all possibilities which they deem consistent with their consideration sets.

### 3.2.3 Epistemic interpretation of blocking

Our framework also lends itself to an epistemic characterization of the blocking notion in Definition 3 which we provide here. The epistemic characterization yields a descriptive formulation of the blocking notion in terms of the agents’
information and knowledge.

Fix a state \((\mu, p, t, \Pi)\) and a potential blocking combination \((i, j; p)\). We define \(B_k\) as the event in which agent \(k\) benefits from being rematched with her or his potential partner, i.e.,

\[
B_i := \{ t' \in T : \nu_{v(i),v(j)} + p > \nu_{v(i),v(\mu(i))} + p_{i,\mu(i)} \}
\]

and

\[
B_j := \{ t' \in T : \phi_{v(i),v(j)} - p > \phi_{v(\mu(j)),v(j)} - p_{\mu(j),j} \}.
\]

If firm \(j\) knows that \(B_j\) does not occur, she knows that she will not benefit from forming the rematching with worker \(i\) at wage \(p\) and hence will not join the blocking combination. That is, conditional on the blocking combination being formed, worker \(i\) could safely ignore any type assignment at which firm \(j\) knows that \(B_j\) does not occur. This refinement can be formulated with worker \(i\) hypothetically updating his partition with the hypothetical information represented by the binary partition \(\{\mathcal{K}_{j,\Pi_j}(B^c_j), T \setminus \mathcal{K}_{j,\Pi_j}(B^c_j)\}\), namely that

\[
\mathcal{U}_i(\Pi_i, \Pi_j) := \Pi_i \vee \{\mathcal{K}_{j,\Pi_j}(B^c_j), T \setminus \mathcal{K}_{j,\Pi_j}(B^c_j)\}.
\]

The knowledge operator is written as \(\mathcal{K}_{j,\Pi_j}\) instead of \(\mathcal{K}_{j,\Pi}\) because we anticipate to hypothetically update \(\Pi_i\) and \(\Pi_j\) but not other agents’ partitions. Likewise, conditional on the blocking combination \((i, j; p)\) being formed, firm \(j\) can also hypothetically update \(\Pi_j\) to

\[
\mathcal{U}_j(\Pi_i, \Pi_j) := \Pi_j \vee \{\mathcal{K}_{i,\Pi_i}(B^c_i), T \setminus \mathcal{K}_{i,\Pi_i}(B^c_i)\}.
\]

For \(k = i, j\), let \(\Pi_k^{l+1} \equiv \mathcal{U}_k^{l+1}(\Pi_i, \Pi_j) := \mathcal{U}_k(\mathcal{U}_k^l(\Pi_i, \Pi_j), \mathcal{U}_k^l(\Pi_i, \Pi_j))\) for each \(l \geq 1\). Since \(T\) is finite and \((\Pi^1_i, \Pi^1_j)\) becomes finer and finer, there must be an integer \(l^*\) such that

\[
(\Pi^l_i, \Pi^l_j) = (\Pi^\infty_i, \Pi^\infty_j) \text{ for every } l \geq l^*.
\]

We call \((\Pi^\infty_i, \Pi^\infty_j)\) the hypothetical information structure for the blocking combination \((i, j; p)\).

Volijs (2000) defines a notion of core under incomplete information which incorporates endogenous communication of willingness to trade among the agents. While the definition of hypothetical information structure may appear similar to the definition of core in Volij (2000), it does not require real communication.
between worker $i$ and firm $j$ about their willingness to join the blocking combination. More precisely, the “information updating” embodied in (10) and (11) builds only upon the “common knowledge” that each agent will rule out the type assignment according to which the other agent never considers joining the blocking combination. Such “common knowledge” must be formulated via refining the agents’ information partition as per (10) and (11) to reflect the type assignments which the agents rule out. The formulation thus makes the hypothetical information structure similar to the information structure resulted from iterative communication of willingness to join the blocking combination. The resulting information structure is however entirely hypothetical as the blocking combination need not be formed.

The following proposition provides an epistemic characterization of blocking in Definition 3:

**Proposition 1** A state $(\mu, p, t, \Pi)$ is blocked by $(i, j; p)$ if and only if under the hypothetical information structure, both $i$ and $j$ know at $t$ that they will benefit from being rematched, i.e., $t \in K_{i,\Pi^\infty}(B_i) \cap K_{j,\Pi^\infty}(B_j)$.

We may compare Proposition 1 with Fact 1. Indeed, individual rationality can be viewed as absence of the blocking combination $(k, k, 0)$. Since the agent learns nothing about her/himself from her/his willingness of staying unmatched, the hypothetical information structure must be identical to the initial information structure $\Pi$.

### 3.3 Stability

Like the notion of stability with complete information, the notion of stability which we are about to propose also requires individual rationality and the absence of blocking. Unlike the notion of stability with complete information, however, our stability must also embody a notion called information stability which is formulated in Chen and Hu (2020) in a setting with one-sided incomplete information and also in Liu et al. (2014) with virtually no account for the firms’
heterogeneous information. The extension to two-sided incomplete information is immediate, as we present as follows.

Define a set of type assignments as follows:

\[ N_{\mu,p,\Pi} := \{ t \in T : (\mu, p, t, \Pi) \text{ is individually rational and unblocked} \} \]

Intuitively, by the public information \((\mu, p, \Pi)\) and the absence of blocking, agents know that the true type assignment lies in \(N_{\mu,p,\Pi}\). Let \(\mathcal{CK}_\Pi\) denote the meet (i.e., finest common coarsening) of the partition profile \(\Pi\). Then, given a state \((\mu, p, t, \Pi)\), the set \(\mathcal{CK}_\Pi(t)\) is the cell of the meet of \(\Pi\) that contains the true type assignment \(t\). Upon observing the absence of blocking, each agent can refine their partitions within \(\mathcal{CK}_\Pi(t)\). We now formally define an operator \(H_{\mu,p}(\cdot)\) to represent the information refinement. For notational convenience, we denote by \(N_{\mu,p,\Pi}\) the binary partition that is induced by \(N_{\mu,p,\Pi}\), i.e., \(N_{\mu,p,\Pi} := \{N_{\mu,p,\Pi}, T \setminus N_{\mu,p,\Pi}\}\).

\[
[H_{\mu,p}(\Pi)]_j(t') := \begin{cases} 
\Pi_j(t') \cap N_{\mu,p,\Pi}(t'), & \text{if } t' \in \mathcal{CK}_\Pi(t); \\
\Pi_j(t'), & \text{otherwise.}
\end{cases}
\] (12)

A state is said to be stable if it is individually rational and unblocked, and moreover, no further information can be inferred from individual rationality and the absence of blocking.

**Definition 4** A state \((\mu, p, t, \Pi)\) is **stable** if it satisfies the following three requirements:

(i) \((\mu, p, t, \Pi)\) is individually rational.
(ii) \((\mu, p, t, \Pi)\) is not blocked.
(iii) \(H_{\mu,p}(\Pi) = \Pi\).

We provide an epistemic foundation for our notion of stability. In particular, stability is characterized by the common knowledge of individual rationality and no blocking.

**Proposition 2** A state \((\mu, p, t, \Pi)\) is stable if and only if at \(t\), it is common knowledge that the state is individually rational and not blocked.
We close this section by the existence of stable states.

**Theorem 1** The set of stable states is nonempty. In particular, for any \( t \in T \), let \((\mu, p)\) be a complete-information stable allocation and \( \Pi \) be an arbitrary information partition profile. Then \((\mu, p, t, H_{\mu,p}(\Pi))\) is stable, where \( H_{\mu,p}(\cdot) \) is the \( l \)-iteration of \( H_{\mu,p}(\cdot) \).

The result says that every complete-information stable allocation is associated with at least one stable state. This fact, together with the existence results of Shapley and Shubik (1971) and Crawford and Knoer (1981), implies that the set of stable states is nonempty. This observation also extends the existence result in Liu et al. (2014) and Chen and Hu (2020) to a setting with two-sided incomplete information. The proof of Theorem 1 will be presented in the Appendix after that of a later result, Proposition 3, which Theorem 1 relies on.

4 Properties of blocking and stable states

In this section, we introduce two properties of blocking, i.e., improvement at the truth and information monotonicity. Both of them are specific to the incomplete-information environment and useful in understanding stability with incomplete information.

4.1 Improvement at the truth

Fix a blocking combination \((i, j; p)\) for the state \((\mu, p, t, \Pi)\). First, we are interested in the essence of blocking—an opportunity to obtain higher payoffs through rematching. Will the two agents \( i \) and \( j \) get *de facto* higher payoffs? If the answer is yes, then we say blocking satisfies improvement at the truth. This is arguably the most basic desideratum behind any blocking notion in the theory of matching. We formalize the property as follows:

**Improvement at the Truth (IT).** If \((\mu, p, t, \Pi)\) is blocked by \((i, j; p)\), then \( \chi_i(t) = Y \) and \( \chi_j(t) = Y \).

We know from the definition of consideration correspondence that at the
true type assignment \( t \), only a subset of \( \Pi_k(t) \) is considered. If IT holds, then \( t \) is considered at \( t \) by both \( i \) and \( j \). More importantly, if IT is satisfied, then blocking with incomplete information implies blocking with complete information. In other words, the set of incomplete-information blocking combinations is a subset of complete-information blocking combinations.

Improvement at the truth holds in the one-sided incomplete-information setting such as Liu et al. (2014) and Chen and Hu (2020). In our more general setup, instead of proving IT directly, we will establish a stronger property in the next subsection which implies IT.

### 4.2 Information monotonicity

We now introduce the second property of blocking as follows:

**Information Monotonicity (IM).** If \((\mu, p, t, \Pi)\) is blocked by \((i, j; p)\) and \(\hat{\Pi}\) is a finer partition profile than \(\Pi\), then \((\mu, p, t, \hat{\Pi})\) is also blocked by \((i, j; p)\).

Clearly, IM is a stronger than IT. More precisely, when \(\hat{\Pi}\) is the complete-information partition profile, \((\mu, p, t, \hat{\Pi})\) is blocked by \((i, j; p)\) if and only if \(\chi_i(t) = Y\) and \(\chi_j(t) = Y\), which is exactly the definition of IT. Intuitively, IM means that when agents have more precise information, it is easier for them to find blocking opportunities.\(^\text{10}\) Indeed, the blocking notion in Definition 3 satisfies IM and, thus, IT.

**Theorem 2** The blocking notion in Definition 3 satisfies IM and IT.

IM is a generalization of Fact 1 in Chen and Hu (2020).

### 4.3 Stable states

Equipping with the properties of blocking, we are now ready to investigate properties of stable states. We first derive an analogy to the IM property of blocking. In particular, we show that if \((\mu, p, t, \hat{\Pi})\) is stable and \(\hat{\Pi}\) is finer than another

\(^{10}\)Another reason for introducing two separating properties is that IM is not satisfied for some variant blocking notions but IT is satisfied. See Section 5.
partition profile $\Pi$, then the state $(\mu, p, t, \Pi)$ is “essentially” stable in the sense that it is not stable only because information stability is not satisfied. In other words, such a state is “essentially” stable in that for every integer $l \geq 0$, the state $(\mu, p, t, H_{\mu, p}^l(\Pi))$ is individually rational and not blocked. The formal result is as follows.

**Proposition 3** Suppose that $\hat{\Pi}$ is a finer partition profile than $\Pi$. If a state $(\mu, p, t, \hat{\Pi})$ is stable, then the state $(\mu, p, t, H_{\mu, p}^\infty(\Pi))$ is stable.

The following theorem says that a matching outcome, i.e., an allocation and a true type profile, arises in some stable state if and only if it induces a particular stable state which has a particular partition profile.

**Theorem 3** Let $\bar{\Pi}$ be the trivial partition profile over $T$ such that agents have the least information, i.e., $\bar{\Pi}_k := \{T\}$ for all $k \in I \cup J$. For an arbitrary matching outcome $(\mu, p, t)$, the state $(\mu, p, t, \Pi)$ is stable for some $\Pi$ if and only if the state $(\mu, p, t, H_{\mu, p}^\infty(\Pi))$ is stable.

The sufficiency part is trivial. The necessity part follows from Proposition 3. This result shows that despite the great variety of incomplete-information situations, the entire set of stable allocations across all information structures can be identified via the set of stable allocations under a specific class of information structures $H_{\mu, p}^\infty(\Pi)$.

5 Alternative blocking and stability notions

In this section, we discuss two alternative blocking notions. In comparison with the blocking notion in Definition 3, the first notion represents a situation where the agents are more conservative in forming a blocking pair, whereas the second notion represents a situation where the agents are more aggressive in forming a blocking pair. Each blocking notion corresponds to a stability notion when associated with individual rationality and information stability. Existence and properties of those stability notions will be discussed in Online Appendix 8.3.
5.1 Naive blocking

Agents may be more conservative in refining their consideration than the way we described in Section 3.2.2. As a result, it is harder for a state to be blocked. An extreme case is that the agents consider all possible type assignments consistent with their information partitions. We say a state is naively blocked if there exist a firm, a worker, and a potential wage such that with the rematching of them, both agents would receive a higher payoff under all type assignments which they deem possible.

**Definition 5** A state \((\mu, p, t, \Pi)\) is naively blocked by \((i, j; p)\) if
\[
\nu_{t'}(i, t') + p > \nu_{t'}(\mu(i)) + p_i,\mu(i) \quad \text{for all } t' \in \Pi_i(t), \text{ and }
\phi_{t'}(i, t') - p > \phi_{t'}(\mu(j)) + p_{\mu(j)},j \quad \text{for all } t' \in \Pi_j(t).
\]

Obviously, if a state is naively blocked, then it is blocked. Naive blocking is equivalent to having the agents in a blocking combination set their consideration set equal to their information partition cell at each type assignment. To understand when the consideration iteration process in Section 3.2.2 has a bite, we provide sufficient conditions under which naive blocking and blocking are equivalent. More precisely, we state the following four conditions:

**A1.** (One-Dimensional Type) \(W \subset \mathbb{R}\) and \(F \subset \mathbb{R}\).

**A2.** (Increasing Utility) The premuneration functions \(\nu_{w,f}\) and \(\phi_{w,f}\) are strictly increasing in \(w\) and \(f\).

**A3.** (Non-Transferable Utility) No transfer is permitted.

**A4.** (Knowledge within One Side) It is common knowledge that each worker knows the types of all workers and each firm knows the types of all firms.

These conditions are primitive and have all appeared in the literature: conditions A1-A2 are imposed in Liu et al. (2014); condition A1-A3 in Bikhchandani (2017); and condition A4 in Pomatto (2019).
Theorem 4 Under conditions A1-A4, a state is naively blocked if and only if it is blocked.\textsuperscript{11}

Conditions A1 and A2 are standard and they hold in many environments. Example 1 illustrates that even if A1-A3 hold simultaneously, without A4, a state can be blocked but not naively blocked. The following example demonstrates the importance of A3 in establishing the equivalence, in which conditions A1, A2 and A4 are satisfied. The examples show that in those environments where A1 and A2 hold, it is in general necessary to distinguish different blocking/stability notions.

Example 2 Consider a market with two workers, $\alpha$ and $\beta$, and one firm, $a$. The set of possible type assignments is given by $T = \{t, t'\}$, i.e.,

\[
\begin{array}{ccc}
t & t' \\
\alpha & 3 & 3 \\
\beta & 4 & 1 \\
a & 2 & 2 \\
\end{array}
\]

The prenumeration values for workers and firms are given by $\nu_{w,f} = \phi_{w,f} = w_f$. Obviously, given any wage, every agent prefers a partner with a higher type to a partner with a lower type.

We first describe a state $(\mu, p, t, \Pi)$. Suppose $\mu(\alpha) = a$, $\mu(\beta) = \beta$, and $p = 0$. Suppose $t$ is the true type assignment, and that agents’ partitions are given as if agents have the knowledge of the agent types within their own side, i.e., $\Pi_\alpha = \Pi_\beta = \{\{t\}, \{t'\}\}$, and $\Pi_a = \{t, t'\}$.

Now consider the potential blocking combination $(a, \beta; -3)$. Using the partitions over lists of indicators as in Example 1, we rewrite agents’ partitions in the left panel below. The consideration correspondences of $a$ and $\beta$ could then be represented as in the right panel.

\textsuperscript{11}Under the same set of conditions, naive blocking and blocking are also equivalent to the notion of sophisticated blocking to be introduced in the next subsection; the claim will be formally stated and proved in the online appendix.
Clearly, \((a, \beta; -3)\) blocks the status quo state, but not naively.

5.2 Sophisticated blocking

Contrary to naive blocking, we can also think of blocking notions in which the agents are more aggressive. In particular, we may conceive scenarios other than the one illustrated in Example 1 in which the agents can refine their consideration further. With more aggressive refinement of consideration within the potential blocking pair, it is easier for a state to be blocked. Recall that in Example 1, the key driver of worker \(\beta\)'s consideration refinement is that there exists a cell of \(\Pi_a\) in which firm \(a\) always objects, i.e., firm \(a\)'s indicators are always \(\bar{N}\).

We give two other scenarios different from the uniform \(\bar{N}\) case to exemplify how the agents may refine their consideration further. First, consider the scenario in the left panel of the figure below. Namely, \(Y\) and \(N\) coexist in some cell of an agent’s partition. At the type assignment whose indicator is \(N\), the potential partner’s indicators are uniformly \(Y\)’s. Because of the potential partner’s uniform \(Y\), at that cell, the potential partner is always willing to join the blocking regardless of her consideration set.\(^{12}\) Thus, plausibly, the agent would always consider the type assignment whose indicator is \(N\). Then the agent would definitely say no, i.e., the undetermined case should become a determined objection. As a result, the potential partner does not consider the first two type assignments. We refer to this scenario as the supported \(\bar{N}\) case.

\[
\begin{array}{c|c}
\text{Agent} & \{Y, N\} & \cdots & \text{Agent} & \{Y, N\} & \cdots \\
\text{Potential partner} & \cdots & \{Y, Y\} & \text{Potential partner} & \{Y, N, Y\}
\end{array}
\]

Now consider the scenario in the right panel of the figure. The agent’s partition cell is contained in her potential partner’s corresponding partition cell.\(^{12}\)

\(^{12}\)As long as the consideration set is a subset of the corresponding partition cell.
Then the agent’s partition cell belongs to the join of her and her potential partner’s partitions. Thus, however the two agents update their consideration, the agent can never distinguish $Y$ and $N$, which also renders the undetermined case a determined objection. As a result, the potential partner does not consider the first two type assignments. We refer to this scenario as the *inseparable* $N$ case. Incorporating all uniform $N$’s, supported $N$’s and inseparable $N$’s into the starting point of the iterative definition of consideration set will lead to, what we call, a *sophisticated blocking*. Intuitively, if a state is blocked, then it is sophisticatedly blocked.

We now elaborate why we specifically choose the *supported* $N$ case and the *inseparable* $N$ case, on top of the *uniform* $N$ case to start the consideration refinement in sophisticated blocking. Suppose worker $i$ and firm $j$ are evaluating a new partnership with each other. In Section 3.2.2, we have introduced a restriction (7) that a general consideration correspondence $C^k_\dagger(\cdot)$ should satisfy, i.e.,

$$C^k_\dagger(t) \subset \Pi_k(t) \text{ for } k = i, j \text{ and all } t \in T.$$  

Here we propose another basic restriction which a general consideration correspondence shall satisfy. It says that each consideration set, when it is nonempty, can be expressed as a union of cells in the join of $\Pi_i$ and $\Pi_j$:  

$$t' \in C^k_\dagger(t) \text{ implies } [\Pi_i \lor \Pi_j](t') \subset C^k_\dagger(t), \text{ for } k = i, j \text{ and all } t, t' \in T. \quad (13)$$

Clearly, (7) restricts the upper bound, whereas (13) restricts the lower bound of consideration sets.

As long as $C^k_\dagger(\cdot)$ satisfies (7) and (13), and agent $j$ has either a supported $N$ or an inseparable $N$ at $t'$, we know that either $C^j_\dagger(t')$ is empty or $C^j_\dagger(t')$ must contain at least one $N$.\(^{13}\) That means agent $i$ can safely rule out $t'$ from the beginning. Hence, the two kinds of $N$ satisfies the same criterion which applies to the uniform $N$ case, whenever the consideration correspondence satisfies (7) and (13).

\(^{13}\)See Online Appendix 8.1 for the explanation of adding nonemptiness and more comprehensive discussions.
Of course, not all scenarios with both $Y$ and $N$ faced by an agent can initiate consideration refinements, as illustrated by the cases below:

Agent $\{Y \ N\}$ ... Agent $\{Y \ N\}$ ...
Potential partner ... $\{Y \ N\}$ Potential partner ... $\{N \ Y\}$

In these two cases, whether the agent can rule out $N$ cannot be deduced from (7) and (13) only and must depend on how the consideration set is defined/refined for both agents. For instance, in the left panel, if the potential partner can rule out her $N$ by, say, uniform $N$’s of the agent, then the potential partner will consider only $Y$ and the agent will have to consider $N$. Alternatively, if the potential partner’s $N$ is supported, say, due to the agent’s uniform $Y$’s, then the potential partner’s consideration set will have to contain the $N$ and the agent can safely rule out his $N$. Either case can happen. In the same vein, in the right panel, whether the agent will definitely consider the $N$ cannot be deduced from (7) and (13) only.

5.3 Properties of IT and IM for blocking notions

Obviously, the naive blocking satisfies IM and thus IT. Unlike the naive blocking, the sophisticated blocking satisfies IT but not IM. Violation of IM is illustrated in Figures 1 and 2, where either supported $N$ or inseparable $N$ cannot help to refine consideration when information becomes finer; a complete description of these examples and detailed discussions can be found in Online Appendix 8.4.

![Figure 1](image)

(a) The original state (b) The state with finer information

Figure 1: Refinement due to inseparable $N$ disappears with finer information.
The properties of IT and IM for different blocking notions are summarized in the table below; see the online appendix for the formal definition of sophisticated blocking, formal discussions of these properties and formal comparison between different blocking/stability notions. To facilitate comparison, we also include the complete-information blocking in the table.

<table>
<thead>
<tr>
<th>Blocking Notions</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IT</td>
</tr>
<tr>
<td>Complete-information blocking</td>
<td>satisfied</td>
</tr>
<tr>
<td>Sophisticated blocking</td>
<td>satisfied</td>
</tr>
<tr>
<td>Blocking</td>
<td>satisfied</td>
</tr>
<tr>
<td>Naive blocking</td>
<td>satisfied</td>
</tr>
</tbody>
</table>

### 5.4 Further consideration refinement

Is it possible for agents to make further inference from the limit consideration sets? To illustrate the issue, consider the situation depicted in the figure below.

$$
\text{Truth} \\
\text{Worker} \quad \{Y \quad N\} \quad \{Y\} \\
\text{Firm} \quad \{Y\} \quad \{Y \quad N\}
$$

In this situation, there are three type assignments $t^1$, $t^2$ and $t^3$. Obviously, there is no uniform $N$, inseparable $N$, or supported $N$. As a result, at $t^1$ or $t^3$, both agents’ consideration set coincides with their partition cell and either the worker or the firm would object the rematching, whether we adopt naive
blocking, blocking, or sophisticated blocking.

Now consider the following reasoning to “justify” a blocking combination for the situation. As the worker cannot distinguish $t^1$ and $t^2$, he may think that, given the limit consideration sets, the firm will object the rematching if the true type profile is $t^2$. Therefore, the worker does not need to consider the type profile $t^2$ when the true type assignment is $t^1$. It follows that the worker (as well as the firm) would be willing to join the rematching at $t^1$, and symmetrically, at $t^3$. However, this “consideration refinement at the limit” is inconsistent. To wit, when the worker rules out $t^2$ at $t^1$, he must assume that the firm will object the rematching at $t^2$. The basis for the latter assumption, however, is that the firm has “confirmed” the $N$ at $t^2$ and hence the decision of not participating in the rematching at type assignment $t^2$ (and hence $t^3$). This assumption is not warranted, if at type assignment $t^3$ (and hence $t^2$), the firm is also following symmetrically the reasoning of the worker at $t^1$ to entertain the possibility of joining the rematching.

This example illustrates why in defining sophisticated blocking, we opt to search for a proper starting point such as the three kinds of $N$ which make no reference to how we define a consideration set and only to the basic properties (7) and (13).

6 Concluding Remarks

In this paper, we provided a general framework for studying two-sided matching markets with incomplete information. We proposed a stability notion as a solution concept for these matching markets and also established its existence and epistemic foundation. We documented properties of the blocking notion and stable states which are specific to the incomplete-information environment. We also examined two alternative blocking/stability notions which represent the two extreme cases of consideration refinement.

We expect more research on stable matching with incomplete information
being conducted based upon this general framework. For instance, it remains to be investigated whether, and to what extent, standard results on the structure of stable matchings hold in the general incomplete-information environments.\textsuperscript{14} There might also be other structures on the set of stable states in terms of the information structure beyond IT and IM.

7 Proofs

Proof of Lemma 1 We prove by induction. Obviously, we have
\[ C_i^1(t') \subset C_i^0(t') = \Pi_i(t') \text{ and } C_j^1(t') \subset C_j^0(t') = \Pi_j(t'). \]
Suppose the claim of the lemma holds for step \( l \geq 1 \), i.e.,
\[ C_i^l(t') \subset C_i^{l-1}(t') \text{ and } C_j^l(t') \subset C_j^{l-1}(t'). \tag{14} \]
We proceed to show that
\[ C_i^{l+1}(t') \subset C_i^l(t') \text{ and } C_j^{l+1}(t') \subset C_j^l(t'). \]
Suppose \( t'' \in C_i^{l+1}(t') \). Then there exists \( t''' \in C_j^l(t'') \) such that \( \chi_j(t''') = Y \). By (14), \( C_j^l(t') \subset C_j^{l-1}(t') \). Therefore, \( t''' \in C_j^{l-1}(t'') \) and \( \chi_j(t''') = Y \). Hence, \( t'' \in C_i^l(t') \) and consequently \( C_i^{l+1}(t') \subset C_i^l(t') \). The argument for \( C_j^{l+1}(t') \subset C_j^l(t') \) is symmetric.

Proof of Proposition 1 We proceed to show that in either direction,
\[ C_i^l(t) = \Pi_i^l(t) \text{ and } C_j^l(t) = \Pi_j^l(t) \text{ for every } l = 0, 1, 2, \ldots, \tag{15} \]
which particularly lead to \( C_i^\infty(t) = \Pi_i^\infty(t) \) and \( C_j^\infty(t) = \Pi_j^\infty(t) \). Since \( \Pi_i^\infty \) and \( \Pi_j^\infty \) are partitions, we know that \( C_i^\infty(t) \neq \emptyset \) and \( C_j^\infty(t) \neq \emptyset \) must hold if (15) holds. Obviously, (8) holds if and only if \( C_i^\infty(t) \subset B_i \). Similarly, (9) holds if and only if \( C_j^\infty(t) \subset B_j \). Therefore, \( C_i^\infty(t) = \Pi_i^\infty(t) \) would imply that \( (\mu, \mathbf{p}, \mathbf{t}, \Pi) \) is blocked by \( (i, j; p) \) if and only if \( \Pi_i^\infty(t) \subset B_i \) and \( \Pi_j^\infty(t) \subset B_j \), which are exactly the same as \( t \in K_i, \Pi_i^\infty(B_i) \) and \( t \in K_j, \Pi_j^\infty(B_j) \) respectively.

To see that (15) holds, we use induction. First of all, we have
\[ C_i^0(t) = \Pi_i = \Pi_i^0(t) \text{ and } C_j^0(t) = \Pi_j = \Pi_j^0(t). \]
\textsuperscript{14}For instance, Example 2 in Chen and Hu (2020) illustrates that the celebrated Rural Hospital Theorem (c.f. Theorems 2.22 and 5.12 of Roth and Sotomayor (1990)) no longer holds with incomplete information.
Suppose (15) holds for step $l \geq 0$. Then
\[
C_{i}^{l+1}(t) = \{t' \in \Pi_i(t) : \exists t'' \in C_j^l(t') \text{ s.t. } \chi_j(t'') = Y\}
\]
\[
= \{t' \in \Pi_i(t) : \exists t'' \in \Pi_j(t') \text{ s.t. } \chi_j(t'') = Y\}
\]
\[
= \Pi_i(t) \cap \left[ T \setminus \mathcal{K}_{j, \Pi_j}(B^c_j) \right],
\]
where the second equality follows from Lemma 1, the third equality follows from the induction hypothesis and fourth equality follows from the definition of $K_j, \Pi_j$.
We claim that $t \in T \setminus \mathcal{K}_{j, \Pi_j}(B^c_j)$, which leads us to $\Pi_i(t) \cap \left[ T \setminus \mathcal{K}_{j, \Pi_j}(B^c_j) \right] = \Pi_{i+1}(t)$, and thus $C_{i}^{l+1}(t) = \Pi_{i+1}(t)$. To see this, suppose $t \in \mathcal{K}_{j, \Pi_j}(B^c_j)$. Then either $t \in \mathcal{K}_{j, \Pi_j}(B^c_j)$ is violated in the “if” direction because $\Pi_j(t) \subset B^c_j$ and $\Pi_j^\infty$ is finer than $\Pi_j$, or (9) is violated in the “only-if” direction because $\Pi_j(t) \subset B^c_j$ and $C^\infty_j(t) \subset C^l_j(t) = \Pi_j(t)$. The argument for $C_{j}^{l+1}(t) = \Pi_{j+1}(t)$ is symmetric. This completes the proof of (15) and thus the proposition. ■

Proof of Proposition 2  Note that $H_{\mu, p}(\Pi) = \Pi$ is equivalent to $\mathcal{C} \mathcal{K}_\Pi(t) \subset N_{\mu, p, \Pi}$. Then
\[
(\mu, p, t, \Pi) \text{ is a stable state } \iff t \in N_{\mu, p, \Pi} \text{ and } H_{\mu, p}(\Pi) = \Pi \implies t \in N_{\mu, p, \Pi} \text{ and } \mathcal{C} \mathcal{K}_\Pi(t) \subset N_{\mu, p, \Pi} \iff \mathcal{C} \mathcal{K}_\Pi(t) \subset N_{\mu, p, \Pi}.
\]
Since $\mathcal{C} \mathcal{K}_\Pi$ is the meet of the partition profile $\Pi$, every cell of it, particularly $\mathcal{C} \mathcal{K}_\Pi(t)$, is a self-evident event. Hence, $\mathcal{C} \mathcal{K}_\Pi(t) \subset N_{\mu, p, \Pi}$ is equivalent to saying that at $t$, it is common knowledge that the state is individually rational and not blocked. ■

Proof of Theorem 2  Throughout this proof, we let $l^*$ be the smallest integer such that $C^l_k = C^\infty_k$, where $k = i, j$. We need two lemmata to prove the theorem. First, we show that consideration correspondences in iteration (5)-(6), particularly the limit ones, must be measurable with respect to $\Pi_i \lor \Pi_j$.\footnote{We say that a set is measurable w.r.t. $\Pi_i \lor \Pi_j$ if it is measurable w.r.t. the $\sigma$-algebra generated by all unions of sets in $\Pi_i \lor \Pi_j$.} Therefore, should $t'$ be the true type profile, the smallest possible consideration set is
Lemma 2 (Measurability.) For each $l$, the correspondences $C^l_i$ and $C^l_j$ are measurable w.r.t. $\Pi_i \lor \Pi_j$.

Proof Pick an arbitrary $t'$ such that $C^1_i(t') \neq \emptyset$. Let $t'' \in C^1_i(t')$. We proceed to show that for all $t''' \in [\Pi_i \lor \Pi_j](t'')$, we must have $t''' \in C^1_i(t')$. This is true because $t''' \in [\Pi_i \lor \Pi_j](t'')$ implies that $\Pi_j(t'') = \Pi_j(t'''')$, i.e., $C^0_j(t'') = C^0_j(t'''')$, and that

$$
\exists t''' \in C^0_j(t''') \text{ s.t. } \chi_j(t''') = Y \iff \exists t''' \in C^0_j(t'''') \text{ s.t. } \chi_j(t'''') = Y,
$$

which precisely implies Therefore, $C^1_i(t')$ is measurable w.r.t. $\Pi_i \lor \Pi_j$. Similar argument applies to $C^1_j(t')$. Induction completes the proof.

Next, we prove a stronger version of IT.

Lemma 3 (Strong Improvement at the Truth (SIT).) If $(\mu, p, t, \Pi)$ is blocked by $(i,j; p)$, then for every $t' \in [\Pi_i \lor \Pi_j](t)$, $\chi_i(t') = Y$ and $\chi_j(t') = Y$.

Proof We prove by contradiction. Suppose that there exists $t' \in [\Pi_i \lor \Pi_j](t)$ such that $\chi_i(t') = N$. Since $(\mu, p, t, \Pi)$ is blocked by $(i,j; p)$, we know that $\chi_i(t''') = Y$ for every $t''' \in C^\ast_i(t)$. Therefore, measurability (Lemma 2) implies

$$
C^\ast_i(t) \cap [\Pi_i \lor \Pi_j](t) = \emptyset. \tag{16}
$$

By the iteration rule (5) and measurability, (16) is true only if one of the following two cases happens (for agent $j$ at round $l^\ast - 1$):

(a) $C^{l^\ast-1}_j(t) \neq \emptyset$ and for every $t''' \in C^{l^\ast-1}_j(t)$, we have $\chi_j(t''') = N$.

(b) $C^{l^\ast-1}_j(t) = \emptyset$, which implies $C^{l^\ast-1}_j(t) \cap [\Pi_i \lor \Pi_j](t) = \emptyset$.

Suppose case (a) holds. Then monotonicity, i.e., $C^\ast_j(t) \subset C^{l^\ast-1}_j(t)$ (Lemma 1), implies that for every $t''' \in C^\ast_j(t)$, we have $\chi_j(t''') = N$, which contradicts $(\mu, p, t, \Pi)$ being blocked by $(i,j; p)$. Suppose case (b) holds. This is true only if either case (a) or case (b) holds for agent $i$ at round $l^\ast - 2$. Similar argument shows that case (a) leads to a contradiction. This continues until round zero is reached and only case (b) is possible, i.e., we have either (case (b) for agent $i$ at
round zero)
\[ C^0_i(t) \cap [\Pi_i \vee \Pi_j](t) = \emptyset \]
or (case (b) for agent \( j \) at round zero)
\[ C^0_j(t) \cap [\Pi_i \vee \Pi_j](t) = \emptyset. \]
However, both of which bring us to a contradiction since \( C^0 = \Pi \). Hence, for all \( t' \in [\Pi_i \vee \Pi_j](t) \), we have \( \chi_i(t') = Y \). Symmetric argument shows that \( \chi_j(t') = Y \) holds as well for all \( t' \in [\Pi_i \vee \Pi_j](t) \).

Now we prove Theorem 2. Suppose \( (\mu, p, t, \Pi) \) is blocked by \((i,j);p\). Let \( \hat{\Pi} \) be finer than \( \Pi \). Then we know that for each \( t' \in T \),
\[ \hat{\Pi}_i(t') \subset \Pi_i(t') \) and \( \hat{\Pi}_j(t') \subset \Pi_j(t') \).
Note that \( C^l_i(t') \) defined in (5) is increasing in both \( \Pi_i(t') \) and \( C^{l-1}_j(t'') \), and that \( C^l_j(t') \) defined in (6) is increasing in both \( \Pi_j(t') \) and \( C^{l-1}_i(t'') \). It follows from induction that for each \( t' \in T \) and each \( l = 1, 2, \ldots \),
\[ \hat{C}^l_i(t') \subset C^*_i(t') \) and \( \hat{C}^l_j(t') \subset C^*_j(t') \).
Particularly, we have that for each \( t' \in T \),
\[ \hat{C}^*_{i'}(t') \subset \hat{C}^*_i(t') \) and \( \hat{C}^*_{j'}(t') \subset \hat{C}^*_j(t') \),
where the dependence of \( l^* \) on \( \hat{\Pi} \) and \( \Pi \) is suppressed in the notation.

Now it suffices to show \( \hat{C}^*_{i'}(t) \neq \emptyset \) and \( \hat{C}^*_{j'}(t) \neq \emptyset \). Since \( (\mu, p, t, \Pi) \) is blocked by \((i,j);p\), we know by SIT (Lemma 3) that
\[ \nu_{t(i),t(j)} + p > \nu_{t(i),t(\mu(i))} + p_{i,\mu(i)} \) and \( \phi_{t(i),t(j)} - p > \phi_{t(\mu(j)),t(j)} + p_{\mu(j),j} \).
Therefore, the true type assignment \( t \) is not eliminated from the consideration sets \( \hat{C}^l_i(t) \) and \( \hat{C}^l_j(t) \) along (5)-(6), for all \( l = 1, 2, \ldots \). In particular, \( t \in \hat{C}^*_{i'}(t) \) and \( t \in \hat{C}^*_{j'}(t) \). Therefore, the limit consideration sets are nonempty for both agents, i.e., \( \hat{C}^*_{i'}(t) \neq \emptyset \) and \( \hat{C}^*_{j'}(t) \neq \emptyset \). It follows from Definition 3 that \( (\mu, p, t, \hat{\Pi}) \) is blocked. ■

Proof of Proposition 3  Suppose \( (\mu, p, t, \hat{\Pi}) \) is stable. Then \( (\mu, p, t, \hat{\Pi}) \) is individually rational and not blocked. Moreover, by the definition of \( \mathcal{N}_{\mu,p,\hat{\Pi}}(t) \), we know that \( (\mu, p, t', \hat{\Pi}) \) is individually rational and not blocked for all \( t' \in \mathcal{N}_{\mu,p,\hat{\Pi}}(t) \). The individual rationality of \( (\mu, p, t', \hat{\Pi}) \) implies the individual ratio-
nality of \((\mu, p, t', \Pi)\) because \(\hat{\Pi}\) is finer than \(\Pi\). By Theorem 2, \((\mu, p, t', \hat{\Pi})\) being not blocked implies that \((\mu, p, t', \Pi)\) is not blocked. Therefore, we have
\[
N_{\mu, p, \hat{\Pi}}(t) \subset N_{\mu, p, \Pi}(t).
\]  
(17)

Since \(\hat{\Pi}_k(t') \subset N_{\mu, p, \hat{\Pi}}(t)\) for every \(t' \in N_{\mu, p, \hat{\Pi}}(t)\) and every \(k\) (by the definition of \(H(\cdot)\) in (12)), it follows from (17) that
\[
\hat{\Pi}_k(t') \subset N_{\mu, p, \Pi}(t)
\]  
for all \(t' \in N_{\mu, p, \hat{\Pi}}(t)\) and all \(k \in I \cup J\).

Since \(\hat{\Pi}\) is finer than \(\Pi\), it follows that
\[
\hat{\Pi}_k(t') \subset \Pi_k(t')
\]  
for all \(t' \in N_{\mu, p, \hat{\Pi}}(t)\) and all \(k \in I \cup J\).

Therefore, for every \(k\) and every \(t' \in N_{\mu, p, \hat{\Pi}}(t)\), the definition of \(H(\cdot)\) in (12) implies that
\[
\hat{\Pi}_k(t') \subset \Pi_1^k(t').
\]  
(18)

We claim that the state \((\mu, p, t', \Pi_1)\) is not blocked for every \(t' \in N_{\mu, p, \hat{\Pi}}(t)\).
Otherwise, the state \((\mu, p, t', \hat{\Pi})\) is blocked by Theorem 2 (applied locally on the event \(N_{\mu, p, \hat{\Pi}}(t)\)), which is a contradiction. Hence, \((\mu, p, t', \Pi_1)\) is not blocked.

The individual rationality of \((\mu, p, t', \hat{\Pi})\) implies the individual rationality of \((\mu, p, t', \Pi_1)\) because \(\hat{\Pi}\) is finer than \(\Pi_1\) on the event \(N_{\mu, p, \hat{\Pi}}(t)\).

Inductively for every integer \(l \geq 1\), we have
\[
N_{\mu, p, \hat{\Pi}}(t) \subset N_{\mu, p, \Pi_l}(t).
\]  
(19)

Thus, for every \(t' \in N_{\mu, p, \hat{\Pi}}(t)\) and every \(k\), we have
\[
\hat{\Pi}_k(t') \subset \Pi_1^k(t'),
\]  
which, together with (19) implies that \((\mu, p, t', \Pi_l)\) is individually rational and not blocked. Particularly, \((\mu, p, t, \Pi_l)\) is individually rational and not blocked. Since \(\Pi_l\) is weakly finer than \(\Pi_{l-1}\), there exists \(l^*\) such that \(\Pi_{l^*+1} = \Pi_{l^*}\). Therefore, the limit state \((\mu, p, t, \Pi_{l^*}) = (\mu, p, t, \Pi_l)\) is stable. 

We need the following lemma to prove Theorem 1. The lemma also serves as a proof for the existence of naive stable states and sophisticated stable states, because it is independent of how blocking is defined. Moreover, for naive stability, the existence statement can be as strong as Theorem 1 since IM is satisfied.
Lemma 4 For any $t \in T$, let $(\mu, p)$ be a complete-information stable allocation and $\Pi$ be the complete-information partition profile. Then $(\mu, p, t, \Pi)$ is stable.

Proof of Lemma 4 For any $t \in T$, let $(\mu, p)$ be a complete-information stable allocation at $t$, which exists by, say, Shapley and Shubik (1971). Define a partition profile $\Pi$ such that $\Pi_k(t') = \{t'\}$ for every $t' \in T$ and every $k$. We proceed to verify that $(\mu, p, t, \Pi)$ is stable. First, since $(\mu, p)$ is a complete-information stable allocation at $t$ and $\Pi$ represents complete information, $(\mu, p, t, \Pi)$ is obviously individually rational.

Second, we claim $(\mu, p, t, \Pi)$ is not blocked. Suppose to the contrary that it is blocked by a combination $(i, j; p)$. Then we have $C_\infty^i(t) \neq \emptyset$ and $C_\infty^j(t) \neq \emptyset$. On the other hand, by Lemma 1, both $C_1^i(t)$ and $C_1^j(t)$ are decreasing in $l$, which implies that $C_\infty^i(t) \subset \Pi_i(t) = \{t\}$ and $C_\infty^j(t) \subset \Pi_j(t) = \{t\}$. Hence, we have $C_\infty^i(t) = \{t\}$ and $C_\infty^j(t) = \{t\}$. As a result, (8)-(9) say that $(\mu, p)$ is complete-information blocked by $(i, j; p)$, a contradiction.

Finally, since $\Pi_k(t) = \{t\}$ for every $t \in T$ and every $k$, i.e., $\Pi$ is already the finest partition profile, we know that $\Pi$ is a fixed point of $H_{\mu, p}(\cdot)$ regardless of $(\mu, p)$. Therefore, the state $(\mu, p, t, \Pi)$ is stable.

Proof of Theorem 1 It follows from Lemma 4 and Proposition 3.

Proof of Theorem 4 We clarify some notations before proving the theorem. Let $t = (w, f)$ and $t' = (w', f')$. Under condition A4, for every $t' \in \Pi_i(t)$, we have $w' = w$ and for every $t' \in \Pi_j(t)$, we have $f' = f$. In this sense, we need to treat $w$ and $f$ separately. Taking condition A3 (no transfer) into account, we shall now write a typical state as $(\mu, w, f, \Pi)$, and a blocking combination now is simply a blocking pair.

Consider a blocking pair $(i, j)$ for the state $(\mu, w, f, \Pi)$. It suffices to show that the two agents’ consideration at $(w, f)$ cannot be refined, i.e., $C_\infty^i(w, f) = \Pi_i(w, f)$ and $C_\infty^j(w, f) = \Pi_j(w, f)$.

We first show $C_1^i(w, f) = \Pi_i(w, f)$ and $C_1^j(w, f) = \Pi_j(w, f)$. IT and A3
imply that
\[ \nu_{w(i),f(j)} > \nu_{w(i),f(\mu(i))} \] and \[ \phi_{w(i),f(j)} > \phi_{w(\mu(j)),f(j)}. \] (20)
Since agents prefer higher types by conditions A1-A2, the second part of (20) implies that
\[ w(i) > w(\mu(j)). \] (21)
For each type assignment \((w', f') \in \Pi_i(w, f)\), since worker \(i\) knows the types of all workers (A4), we have \(w' = w\). Thus, for each \((w', f') \in \Pi_i(w, f)\), inequation (21) and conditions A1-A2 imply that
\[ \phi_{w'(i),f'(j)} > \phi_{w(\mu(j)),f'(j)}, \]
which is equivalent to saying that \(\chi_j(w', f') = Y\). Obviously,
\[ (w', f') \in C^0_j(w', f') = \Pi_j(w', f'). \] (22)
Hence, we have \((w', f') \in C^1_i(w, f)\), which implies that \(C^1_i(w, f) = \Pi_i(w, f)\). By symmetric arguments, we know that \(C^1_j(w, f) = \Pi_j(w, f)\).

The arguments for the induction step \(l = 2, 3, \ldots\) is the same as those for the first step except that for each \((w', f') \in C^l_i(w, f) = \Pi_i(w, f)\), we need to replace (22) with the following:
\[ (w', f') \in C^{l-1}_j(w', f') \text{ because of the first part of (20) and } (w, f) \in \Pi_i(w', f') = C^{l-2}_i(w', f'). \]
As a result, \((w', f') \in C^l_i(w, f)\) and \(C^l_i(w, f) = C^{l-1}_i(w, f) = \Pi_i(w, f)\).

Hence, by induction in \(l\), we know that for every \(l\),
\[ C^l_i(w, f) = \Pi_i(w, f) \] and \(C^l_j(w, f) = \Pi_j(w, f) \) (23)
It follows that (23) also holds for the limit consideration correspondences. This completes the proof. ■

References


8 For online publication:

Formal analysis for Section 5

In this online appendix, we formally state and prove the results that are briefly discussed in Section 5. Throughout, we fix a potential blocking combination \((i, j; p)\) and let the partition profile be \(\Pi\).

First, in Appendix 8.1, we formally elaborate why we shall pay specific attention to the \textit{uniform} \(N\) case, the \textit{supported} \(N\) case and the \textit{inseparable} \(N\) case, which we will refer as the fundamental sources of consideration refinements. Then we formally define sophisticated blocking in Appendix 8.2. Related results will be stated in Appendix 8.3 and proved in Appendix 8.4.

8.1 Fundamental sources of consideration refinements

Suppose for now that we do not have a formulation of consideration refinement yet, and we are about to define some consideration correspondences that specifies for each agent at each type assignment a set of considered type assignments. Denoted the consideration correspondences to be defined by \(C^i_\dagger(t)\) and \(C^j_\dagger(t)\), where \(t \in T\).

For convenience, here we restate the upper-bound constraint (7) and the lower-bound constraint (13) introduced in Section 5.2:

\[
C^i_\dagger(t) \subset \Pi_i(t) \quad \text{and} \quad C^j_\dagger(t) \subset \Pi_j(t), \quad \text{for all } t \in T. \quad (7)
\]

\[
t' \in C^i_\dagger(t) \text{ implies } [\Pi_i \lor \Pi_j](t') \subset C^i_\dagger(t), \quad \text{for } k = i, j \text{ and all } t, t' \in T. \quad (13)
\]

Based on these natural restrictions and upon nothing else, we proceed to figure out in which cases worker \(i\) (firm \(j\)) can refine his (her) consideration. Indeed, taking into account the possibility that refinements may lead to further refinements, we shall focus on cases which initiate consideration refinements, instead of defining \(C^i_\dagger(t)\) and \(C^j_\dagger(t)\) directly. And the initiating cases will actually pin down \(C^i_\dagger(t)\) and \(C^j_\dagger(t)\).

We now identify all fundamental sources of consideration refinements, which
could serve as the starting points of an iteration process, like (5)-(6), that will precisely define the consideration correspondences $C^i_\dagger$ and $C^j_\dagger$. Given the symmetry between $i$ and $j$, we will focus on worker $i$’s point of view: When should worker $i$ exclude $t \in \Pi_i(t)$ from consideration? As has been discussed, we shall classify the cases only according to properties (7) and (13) and, of course, common knowledge of the model.

Consider the following cases:

1. For every $t' \in \Pi_j(t)$, we have $\chi_j(t') = N$.\footnote{Recall that the indicator functions $\chi_i$ and $\chi_j$ are defined by (3)-(4) in Section 3.2.}

   In this case, whatever $C^j_\dagger(t)$ is, $C^j_\dagger(t)$ can only be empty or contain just $N$’s by (7) (abusing terminology). The former subcase, as well as similar situations below with empty $C^j_\dagger(t)$, involves more properties of $C^j_\dagger$ other than (7) and (13), which makes the case not fundamental. In the latter subcase, agent $i$ should exclude $t$. This is the source of uniform $N$’s.

2. For every $t' \in \Pi_j(t)$, we have $\chi_j(t') = Y$.

   Now, $C^j_\dagger(t)$ is empty or contains only $Y$’s by (7). The former subcase is not fundamental. In the latter subcase, agent $i$ has no hope to exclude $t$ from consideration, and, in fact, has to consider $t$.

3. For some $t' \in \Pi_j(t)$, we have $\chi_j(t') = N$; and for some $t'' \in \Pi_j(t)$, we have $\chi_j(t'') = Y$.

   Consider the following mutually exclusive subcases:

   a. There is $t''' \in \Pi_j(t)$ such that $\chi_j(t''') \neq \chi_j(t)$ and $t''' \in \Pi_i(t)$.

      Whatever $C^j_\dagger(t)$ is, either it does not contain $t$ (not a fundamental case because it involves further discussion of $C^j_\dagger$ other than (7) and (13)) or we have $t, t''' \in C^j_\dagger(t)$ (this is true by (13) and $t, t''' \in [\Pi_i \lor \Pi_j](t)$).

      In the latter subcase, firm $j$ will definitely object the new partnership, and agent $i$ should ignore $t$. This is the source of inseparable $N$’s.

   b. For any $t''' \in \Pi_j(t)$ such that $\chi_j(t''') \neq \chi_j(t)$, we have $t''' \notin \Pi_i(t)$.
For worker $i$ to ignore $t$, we must have $C'_j(t)$ containing at least one $N$. Pick an arbitrary $t' \in \Pi_j(t)$ such that $\chi_j(t') = N$. Consider the following cases which may result in $t' \in C'_j(t)$:

i. All $Y$’s (and maybe some $N$’s) will be ignored by agent $j$, so that $C'_j(t)$ could at most contain only $N$’s.

This is not a fundamental case.

ii. For every $t'' \in \Pi_i(t')$, we have $\chi_i(t'') = Y$.

In this case, whatever $C'_i(t')$ is, either it is empty (not a fundamental case) or it must contain only $Y$’s by (7). Suppose the latter happens. Then, firm $j$ will have to consider $t'$, i.e., $t' \in C'_j(t)$, due to worker $i$’s definite willingness to participate in the new partnership. This is the source of supported $N$’s.

iii. For every $t'' \in \Pi_i(t')$, we have $\chi_i(t'') = N$.

In this case, whatever $C'_i(t')$ is, either it is empty (not a fundamental case) or it must contain only $N$’s by (7). Suppose the latter happens. Then, firm $j$ will not consider $t'$, i.e., $t' \notin C'_j(t)$.

iv. For some $t'' \in \Pi_i(t')$, we have $\chi_i(t'') = Y$; and for some $t''' \in \Pi_i(t')$, we have $\chi_i(t''') = N$.

Now we are faced with another question of what $C'_i(t')$ shall be, just as we started with. Naturally, instead of starting another round of discussion, we shall stop here and define $C'_i$ and $C'_j$ iteratively using the fundamental sources of consideration refinement which we just identified.

To sum up, there are only three fundamental sources of consideration refinements, uniform $N$’s, inseparable $N$’s and supported $N$’s, exactly as we have introduced. Within them, uniform $N$’s are of order zero in the sense that it is identified using just the information of $\Pi_j$; while inseparable $N$’s and supported $N$’s are of order one in the sense that they are identified using both the information of $\Pi_j$ and the information of $\Pi_i$. Clearly, there is no need to consider any higher order due to the two-sided structure of the model.
8.2 Definition of sophisticated blocking

To define sophisticated blocking, we first describe how agents’ sophisticated consideration correspondences are formed.

We first demonstrate that agents’ willingness/unwillingness (i.e., the indicator functions defined in (3)-(4)) may be adjusted to reflect inseparable $N$’s or supported $N$’s, which helps us to build the discussion upon our uniform-$N$ benchmark in Section 3.2.2. To wit, consider the inseparable $N$ (right panel) and the supported $N$ (left panel) in the figure below:

![Diagram showing agent considerations](image)

In the left panel, since the agent will either consider neither of $Y$ and $N$, or definitely consider the supported $N$, the potential partner can treat the agent’s $Y$ as an $N$ because it is tied with a supported $N$ in the agent’s consideration. In the right panel, since the agent will either consider neither of $Y$ and $N$, or definitely consider both together, again, the potential partner can treat the agent’s $Y$ as an $N$ because it is tied with an inseparable $N$ in the agent’s consideration. Therefore, in terms of consideration refinement, the information conveyed by the inseparable $N$ and the supported $N$ above can be reflected in the following figure, where the hypothetical willingness/unwillingness of the agent is adjusted:

![Diagram showing adjusted agent considerations](image)

These adjustments turn the two kinds of sophisticated $N$’s into the familiar uniform $N$, with which we could build our analysis on the benchmark in Section 3.2.2.

Naturally, as in Section 3.2.2, an agent only considers type assignments such that the potential partner does not have uniform $N$ (up to adjustment). Set $\chi^0 = \chi$, where $\chi$ is defined in (3)-(4), and $C^0 = \Pi$. Agents’ consideration correspondences are defined as the limit of $C_k^l$ in the following double-iteration process:
Indicator function adjustment. For all \( l \geq 1 \) and all \( t' \in T \),
\[
\chi^l_i(t') := \begin{cases} 
N & \text{if either } \exists t'' \in C_i^{l-1}(t') \text{ s.t. } \chi_i^{l-1}(t'') \neq \chi_i^{l-1}(t') \text{ and } t', t'' \in C_i^{l-1}(t') \\
\chi_i^{l-1}(t') & \text{otherwise;} \\
N & \text{if either } \exists t'' \in C_j^{l-1}(t') \text{ s.t. } \chi_j^{l-1}(t'') \neq \chi_j^{l-1}(t') \text{ and } t', t'' \in C_j^{l-1}(t') \\
\chi_j^{l-1}(t') & \text{otherwise.}
\end{cases}
\]

Sophisticated consideration refinement. For all \( l \geq 1 \) and all \( t' \in T \),
\[
C_i^l(t') := \{ t'' \in \Pi_i(t') : \exists t''' \in C_i^{l-1}(t'') \text{ s.t. } \chi_i^{l-1}(t''') = Y \} 
\tag{24}
\]
\[
C_j^l(t') := \{ t'' \in \Pi_j(t') : \exists t''' \in C_j^{l-1}(t'') \text{ s.t. } \chi_j^{l-1}(t''') = Y \} . 
\tag{25}
\]

The following lemma verifies that the consideration correspondence is monotonically decreasing in \( l \).

**Lemma 5** For \( k = i, j \) and each \( t' \in T \), \( C_i^l(t') \) is decreasing in \( l \) w.r.t. set inclusion.

Since the set \( T \) is finite, there exists some \( l^* \) such that \( C_i^l(t') = C_i^{l^*}(t') \) and \( C_j^l(t') = C_j^{l^*}(t') \) for all \( l \geq l^* \) and all \( t' \in T \). We say a state is sophisticatedly blocked by a combination \((i, j; p)\) if both agents in it could have higher rematching payoffs under every type assignment that is sophisticatedly considered at the true one.

**Definition 6** A state \((\mu, p, t, \Pi)\) is sophisticatedly blocked by \((i, j; p)\) if \( C_i^{l^*}(t) \neq \emptyset \), \( C_j^{l^*}(t) \neq \emptyset \) and
\[
\nu_{(i), j} + \nu_{(i), \mu(i)}(i) + p_{i, \mu(i)} \text{ for all } t' \in C_i^{l^*}(t) \text{ and } \phi_{(i), j} + \phi_{(i), \mu(j)}(j) - p_{\mu(j), j} \text{ for all } t' \in C_j^{l^*}(t).
\]

### 8.3 Additional Results for Section 5

In this subsection, we list properties and connections of different blocking and stability notions. To facilitate comparison, we also include the complete-information
blocking/stability and the results already discussed in Section 5 sometimes.

The following proposition examines IT and IM for all blocking notions.

**Proposition 4** *IT and IM of blocking notions are summarized as follows:*

<table>
<thead>
<tr>
<th>Blocking Notions</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete-information blocking</td>
<td>satisfied, not applicable</td>
</tr>
<tr>
<td>Sophisticated blocking</td>
<td>satisfied, not satisfied</td>
</tr>
<tr>
<td>Blocking</td>
<td>satisfied, satisfied</td>
</tr>
<tr>
<td>Naive blocking</td>
<td>satisfied, satisfied</td>
</tr>
</tbody>
</table>

The following proposition ranks blocking notions.

**Proposition 5** *The following statements are true:*

(i) If a state is naively blocked, then it is blocked.
(ii) If a state is blocked, then it is sophisticatedly blocked.
(iii) If a state is sophisticatedly blocked, then it is complete-information blocked.

Moreover, none of the converse is true.

Denote by $B$ the set of blocking combinations. Then an immediate corollary of Proposition 5 is the following: Fix an arbitrary state, we have

$$B^{\text{naive}} \subset B^{\text{blocking}} \subset B^{\text{sophisticated}} \subset B^{\text{complete-information}}.$$

For each of the blocking notions, we have a corresponding stability notion. More precisely, for naive blocking and sophisticated blocking, we take individual rationality as in Definition 1. The way to formulate information stability for all blocking notions is the identical to that of Section 3.3 up to notional replacement. The following proposition says that the set of stable states is nonempty for each of the stability notions.

**Proposition 6** *The sets of naively and sophisticatedly stable states are both nonempty. Particularly, for any $t \in T$, let $(\mu, p)$ be a complete-information stable allocation. Then*
(i) for any partition profile $\Pi$, $(\mu, p, t, H_{\mu, p}^\infty(\Pi))$ is naively stable; and

(ii) for the complete-information partition profile $\Pi$, $(\mu, p, t, \Pi)$ is sophisticatedly stable.

Now we are ready to rank the stability notions.

**Proposition 7** The following statements are true:

(i) A state is naively stable if it is stable.

(ii) A state is stable if it is sophisticatedly stable.

(iii) A state is sophisticatedly stable if it is complete-information stable.

Moreover, none of the converse is true.

Denote by $S$ the set of stable states. Then an immediate corollary of Proposition 7 is the following set-inclusion relation:

$$S^{\text{naive}} \supset S^{\text{blocking}} \supset S^{\text{sophisticate}} \supset S^{\text{complete-information}}.$$

We close this subsection by establishing the equivalence between naive blocking and sophisticated blocking, which will imply the equivalence with blocking by Theorem 4. Equivalence between blocking notions implies equivalence between stability notions. The conditions to guarantee equivalence are exactly the ones A1-A4 we introduced in Section 5.1.

**Proposition 8** Under A1-A4, the following statements are equivalent:

(i) $(\mu, w, f, \Pi)$ is naively blocked.

(ii) $(\mu, w, f, \Pi)$ is blocked.

(iii) $(\mu, w, f, \Pi)$ is sophisticatedly blocked.

An immediate corollary of Proposition 8 is that under A1-A4, the three stability notions are all equivalent.

**8.4 Proofs of Propositions 4-8**

**Proof of Proposition 4** We only need to prove (i) sophisticated blocking satisfies IT and (ii) it does not satisfy IM.
(i) We prove SIT, i.e., \( \chi_i(t') = Y \) and \( \chi_j(t') = Y \) for all \( t' \in [\Pi_i \lor \Pi_j](t) \), by contradiction. The proof will be similar to that of Lemma 3, which uses measurability (Lemma 2). We omit the establishment of measurability in the current context and refer directly to Lemma 2, as the extension is straightforward without changing the statement. However, we present the rest of the proof completely here, instead of just discussing the difference, to avoid confusion.

Suppose that there exists \( t' \in [\Pi_i \lor \Pi_j](t) \) such that \( \chi_i(t') = N \). Since \((\mu, p, t, \Pi)\) is blocked by \((i, j; p)\), we know that 

\[
\chi_i(t'') = Y \text{ for every } t'' \in C^t_i(t).
\]

Therefore, measurability (Lemma 2) implies

\[
C^t_i(t) \cap [\Pi_i \lor \Pi_j](t) = \emptyset.
\]  

(26)

By the iteration of consideration (24)-(25), (26) is true only if one of the following two cases happens (for agent \( j \) at round \( l^* - 1 \)):

(a) \( C^{l^* - 1}_j(t) \neq \emptyset \) and for every \( t''' \in C^{l^* - 1}_j(t) \), we have \( \chi_j^{l^*}(t''') = N \).

(b) \( C^{l^* - 1}_j(t) = \emptyset \), which implies \( C^{l^* - 1}_j(t) \cap [\Pi_i \lor \Pi_j](t) = \emptyset \).

Suppose case (a) holds. Then \( C^t_j(t) \subset C^{l^* - 1}_j(t) \) (Lemma 5), implies that

\[
\chi_j^{l^*}(t''') = N \text{ for every } t''' \in C^{l^*}_j(t).
\]

Since \((\mu, p, t, \Pi)\) is blocked by \((i, j; p)\), we have

\[
\chi_j(t''') = Y \text{ for every } t''' \in C^{l^*}_j(t).
\]

Therefore, these indicators \( Y \)'s under \( \chi_j \) are adjusted to \( N \)'s when we update the indicator functions. Measurability (Lemma 2) implies that these \( Y \)'s never turn to \( N \)'s by inseparable \( N \). Hence, these \( Y \)'s are adjusted to \( N \)'s by some supported \( N \), i.e., for some \( l \), \( C^l_j(t) \) contains some type assignment \( t''' \) with \( \chi_j(t''') = N \) and it is supported by worker \( i \)'s uniform \( Y \)'s. Obviously, such a type assignment \( t''' \) must be considered by firm \( j \) as long as \( C^l_j(t) \neq \emptyset \). Then we have \( t''' \in C^{l^*}_j(t) \). However, this contradicts \( \chi_j(t''') = Y \) for every \( t''' \in C^{l^*}_j(t) \). Therefore, case (a) does not hold.

\(^{17}\)It is generally true that supported \( N \) type assignment must be considered as long as the corresponding consideration set is nonempty.
Suppose case (b) holds. This is true only if either case (a) or case (b) holds for agent $i$ at round $l^* - 2$. Similar argument as in the last paragraph shows that case (a) leads to a contradiction. This continues until round zero is reached and only case (b) is possible, i.e., we have either (case (b) for agent $i$ at round zero) 
\[ C_i^0(t) \cap [\Pi_i \lor \Pi_j] (t) = \emptyset \]
or (case (b) for agent $j$ at round zero) 
\[ C_j^0(t) \cap [\Pi_i \lor \Pi_j] (t) = \emptyset. \]
However, both of them bring us to a contradiction since $C^0 = \Pi$. Hence, for all $t' \in [\Pi_i \lor \Pi_j] (t)$, we have $\chi_i(t') = Y$. Symmetric argument shows that $\chi_j(t') = Y$ holds as well for all $t' \in [\Pi_i \lor \Pi_j] (t)$.

(ii) We provide two counterexamples. In the first one, IM is not satisfied because the consideration refinement due to inseparable $N$ disappears when information becomes finer. In the second one, IM is not satisfied because the consideration refinement due to supported $N$ disappears when information becomes finer. See Examples 3-4 below. \[ \Box \]

**Example 3** Consider a potential blocking pair for a state, where there are four possible type assignments and the agents’ hypothetical willingness/unwillingness is summarized in the figure below.\[^{18}\]

\[
\begin{array}{c|cccc}
\text{Truth} & t^1 & t^2 & t^3 & t^4 \\
\hline
\text{Worker} & \{Y\} & \{N \ Y \ Y\} \\
\text{Firm} & \{Y \ N\} & \{Y \ N\}
\end{array}
\]

Clearly, the firm’s right partition cell, which contains $t^3$ and $t^4$, is a case of inseparable $N$. The firm knows that the true type assignment is either $t^1$ or $t^2$, and she worries about $t^2$, under which she would not obtain a higher payoff through rematching. However, she would think that if the true type assignment is $t^2$, then the worker would know his right partition cell $\{t^2, t^3, t^4\}$, where $t^3$

\[^{18}\]One can easily come up with comprehensive description of the market and the status quo state such that the situation is exactly what we present in the figure. But here we omit those details to only focus on crucial items.
and $t^4$ should not be considered due to the inseparable $N$ of the firm. Therefore, the firm does not need to consider $t^2$, which means that the potential blocking pair is indeed a blocking pair.

Now let us consider an alternative situation in the figure below, where only the partition profile changes.

**Truth**

\[
\begin{array}{cccc}
  t^1 & t^2 & t^3 & t^4 \\
Worker & \{Y\} & \{N\} & \{Y\} \\
Firm & \{Y\} & \{Y\} & \{N\} \\
\end{array}
\]

Clearly, agents have more precise information as both agents’ partitions become strictly finer. However, the worker and the firm no longer constitute a blocking pair because the firm would consider $t^2$ when evaluating the rematching.

**Example 4** Consider a potential blocking pair for a state, where there are three possible type assignments and the agents’ hypothetical willingness/unwillingness is summarized in the figure below.

**Truth**

\[
\begin{array}{ccc}
  t^1 & t^2 & t^3 \\
Worker & \{Y\} & \{N\} & \{Y\} \\
Firm & \{Y\} & \{Y\} & \{N\} \\
\end{array}
\]

Clearly, the firm’s right partition cell, which contains $t^2$ and $t^3$, is a case of supported $N$. The worker knows that the true type assignment is either $t^1$ or $t^2$, and he worries about $t^2$, under which he would not obtain a higher payoff through rematching. However, he would think that if the true type assignment is $t^2$, then the firm would know her right partition cell $\{t^2, t^3\}$, where the indicators should be adjusted into uniform $N$’s due to the supported $N$. Therefore, the worker does not need to consider $t^2$, which means that the potential blocking pair is indeed a blocking pair.

Now let us consider an alternative situation in the figure below, where only the firm’s partition changes.
Truth

<table>
<thead>
<tr>
<th>t¹</th>
<th>t²</th>
<th>t³</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worker</td>
<td>{Y, N}</td>
<td>{Y}</td>
</tr>
<tr>
<td>Firm</td>
<td>{Y}</td>
<td>{Y}</td>
</tr>
</tbody>
</table>

Clearly, agents have more precise information as the firm’s partition becomes strictly finer. However, the worker and the firm no longer constitute a blocking pair because the worker would consider t² when evaluating the rematching.

**Proof of Proposition 5** Straightforward by definitions. ■

**Proof of Proposition 6** Identical to that of Theorem 1 for naively stable states; and identical to that of Lemma 4 for sophisticatedly stable states. ■

**Proof of Proposition 7** Straightforward by definitions. ■

**Proof of Proposition 8** By Theorem 4, it suffices to show that (i, j) is a blocking pair for the state (µ, w, f, Π) if and only if it is a sophisticated one. The only-if part follows from Proposition 5. We proceed to show that if (i, j) is a sophisticated blocking pair, then it is a blocking pair.

First of all, we claim that under A4, there does not exist inseparable N. More precisely, by A4, the join partition must has a singleton cell at every type assignment, particularly at the true type profile (w, f), i.e.,

\[ [\Pi_i \lor \Pi_j](w', f') = \{(w', f')\} \text{ for all } (w', f') \in T. \tag{27} \]

Then, an immediate implication is that there does not exist inseparable N when the two agents update their consideration.

Now we are ready to show that if (i, j) is a sophisticated blocking pair, then it is a blocking pair. The two blocking notion differs only in the refinement of agents’ consideration. Namely, sophisticated blocking takes inseparable N and supported N into account but blocking does not. Since inseparable N never happens, we only need to show that supported N, if not a uniform N, cannot differentiate the two blocking notions (of course, uniform N’s cannot distinguish
the two blocking notions).

Suppose \( C_i^l(w, f) \) can be refined due to agent \( j \)'s supported \( N \). More precisely, suppose that for some \( l \), there exists \( (w', f') \in C_i^l(w, f) \setminus C_i^{l+1}(w, f) \) and some \( (w'', f'') \in C_j^l(w', f') \) such that \( \chi_j^l(w'', f'') = N \) and
\[
\chi_i^l(w'', f'') = Y \quad \text{for all} \quad (w'', f'') \in C_i^l(w'', f'').
\] (28)

By the assumption that we are in a supported \( N \) case which is not a uniform \( N \), \( C_j^l(w', f') \) contains both \( Y \) and \( N \).

We claim that \( (w'', f'') \in C_i^l(w'', f'') \). Otherwise, it was either ruled out by uniform \( N \) or ruled out by supported \( N \), the latter of which implies that the indicators in \( C_j^l(w', f') \) were all adjusted to \( N \). Then both cases contradicts \( C_j^l(w', f') \) containing both \( Y \) and \( N \). Hence, \( (w'', f'') \in C_i^l(w'', f'') \).

Obviously, by (28), \( (w'', f'') \in C_i^l(w'', f'') \) implies \( \chi_i^l(w'', f'') = Y \), which in turn implies by conditions A1-A3 that
\[
f''(j) > f''(\mu(i)).
\]

By A4, firm \( j \) knows the types of firms. Then, for each \( (w''', f''') \in \Pi_j(w''', f''') \), we have \( f''' = f'' \). Since \( (w'', f'') \in C_j^l(w', f') \), we know that \( (w', f') \in \Pi_j(w'', f'') \). As a result, \( f' = f'' \). Therefore, \( \chi_i^l(w', f') = Y \). Since \( (w', f') \in C_i^l(w, f) \setminus C_i^{l+1}(w, f) \) is taken arbitrarily, we know that agent \( j \)'s supported \( N \) only rules out \( Y \) in \( C_i^l(w, f) \).

By symmetric arguments, agent \( i \)'s supported \( N \) only rules out \( Y \) in \( C_j^l(w, f) \).

Since \((i, j)\) is a sophisticated blocking pair for \((\mu, w, f, \Pi)\), we know that with more \( Y \)'s at \((w, f)\) for both agents, \((i, j)\) is also a blocking pair. 

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