Convergence to Bayesian Stable States without Assuming Observability*

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March 10, 2024

Chen and Hu (2024) proposes a concept of Bayesian stability for matching with incomplete information. They also invoke the result of Chen and Hu (2020) to provide a foundation for their solution concept. Particularly, **assuming** in job matching that firms can *observe* the type of their employees, then starting from an arbitrary market state, a random learning-blocking path converges to a Bayesian stable state with probability one. In this companion note, we document how one could prove the following result *without* assuming *observability*, which strengthens Corollary 1 in Chen and Hu (2024).

Proposition. *Suppose payments permitted in the market are integers. Then starting from an arbitrary initial market state,*

(1) *there exists a finite learning-blocking path that leads to a Bayesian stable state.*

(2) *the random learning-blocking path converges with probability one to a Bayesian stable state.*

We omit the model setup and the formulation of market states and learning-blocking paths, which are identical as in the aforementioned papers (up to notation replacement). However, the proof we present below are rigorous and comprehensive.

*This is a companion note for the paper Chen and Hu (2024).

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The **Proposition** is proved by modifying the proof of Theorem 2 in Chen and Hu (2020) (henceforth, CH20). For statement (1), the proving idea is to either (a) construct an increasing sequence of subsets of agents, where each subset is internally stable in the sense that it does not contain a blocking pair, or (b) identify a strict refinement of some information partition. Since both the number of agents and and the number of type profiles are finite, the internally stable set cannot be enlarged indefinitely and the information partitions cannot be refined indefinitely. Hence, the construction must stop after finitely many steps, at a Bayesian stable state. Statement (2) is implied by statement (1); see the argument on page 43 of CH20.

We highlight the differences from CH20's proof before presenting the details. Particularly, here we (i) adopt the *Bayesian* setting and (ii) drop the *observability* assumption. In a Bayesian setting without observability, if a worker and a firm were matched with each getting a strictly positive *expected* payoff, but both of them are currently unmatched, then they may or *may not* be willing to form a blocking coalition to recover the previous partnership. The reason is that the firm's belief may be *updated*, under which she may no longer expect a positive payoff. This phenomenon is new but we argue that being unwilling to recover the previous relationship indicates strict information refinement of the firm, which still fits into case (b) of the idea in CH20.

Moreover, in attempting to conduct the routine construction of a learning-blocking path, CH20 show that, if some "unexpected" blocking pair (Case 3 on page 46) emerges, it must be the case that the firm *has not known* the precise type of the worker; then by matching with the worker, she can observe his type and thus strictly refine her information. Without observability, we show that, in an "unexpected" blocking (Case 3 on page 5), the firm's willingness to get rematched *must be attributed to strictly more refined information* via the worker's (hypothetical) willingness.

In addition to these two key arguments, there are a few minor modifications to CH20's proof. In what follows, we highlight all arguments that are essentially different from CH20 in color purple. For expositional simplicity, we omit the realized type profile t and the prefix "Bayesian", which are fixed throughout the proof.

Proof of the Proposition

The proof of convergence for statement (1) starts with the following simple lemma, which corresponds to Lemma 1 in CH20 but requires a different proof (without relying on observability). Statement (2) is a corollary of statement (1); see page 43 of CH20.

Lemma 1. *Suppose that a state $(\mu, \mathbf{p}, \Pi)$ admits no blocking pair. Then, there exists a finite learning-blocking path to either (i) a stable state or (ii) a state which admits a blocking pair and has a partition profile that is strictly finer than* Π.

Proof. Let $\Pi^0 := \Pi$ and $\Pi^k := H_{\mu,\mathbf{p}}(\Pi^{k-1})$ for every $k \geq 1$. Observe that Π^k is increasingly (weakly) finer in k. For each $k \geq 1$, there are three mutually exclusive cases which may happen: (i) $\Pi^k = \Pi^{k-1}$, (ii) $\Pi^k \neq \Pi^{k-1}$ and the state $(\mu, \mathbf{p}, \Pi^k)$ is blocked, and (iii) $\Pi^k \neq \Pi^{k-1}$ and the state $(\mu, \mathbf{p}, \Pi^k)$ is not blocked.

Since our starting point is a state $(\mu, \mathbf{p}, \Pi)$ that is not blocked, once case (i) happens, we must have a stable state. Then we are done according to the statement of the lemma. Similarly, once case (ii) happen, we are done. Therefore, the only possible reason for the lemma to fail is that case (iii) happens repeatedly, where each time the information structure Π^{k-1} is *strictly* refined to Π^k. However, since we have only finitely many firms and each firm's partition is over the finite set T, the firms' partition profile cannot be strictly refined indefinitely. Hence, case (iii) can be triggered only finitely many times before reaching either case (i) or case (ii). $\square$

To provide a constructive proof of the **Proposition**, we consider two cases: the initial state, denoted by $(\mu^0, \mathbf{p}^0, \Pi^0)$, admits a blocking pair or otherwise. If the initial state admits no blocking pair, then by Lemma 1, there is a finite learning-blocking path to either a stable state or a state with a blocking pair and a partition profile strictly finer than Π^0. Hence, we only need to focus on the case where a state admits a blocking pair.

Definition 1. *A set of agents $A \subset I \cup J$ is **internally stable** under state $(\mu, \mathbf{p}, \Pi)$ if the following hold:*

(i) *Agents in A are only matched with agents in A.*

(ii) *The set A does not contain two agents who form a blocking pair for $(\mu, \mathbf{p}, \Pi)$.*

(iii) *Moreover, every* matched *agent in A has a strictly positive* *expected* *payoff.*

Pick an internally stable set A at state $(\mu^0, \mathbf{p}^0, \Pi^0)$. If $(\mu^0, \mathbf{p}^0, \Pi^0)$ admits a blocking pair $(\bar{i}, \bar{j})$, then either one or both of agents $\bar{i}$ and $\bar{j}$ are outside A. We deal with the case where exactly one agent is in A in Lemma 2 below and then the other case in the main proof of the **Proposition**. To be precise, we show that we can construct a finite learning-blocking path which leads to a state $(\overline{\mu}, \overline{\mathbf{p}}, \overline{\Pi})$ under which either (a) we obtain a strict superset of A which is still internally stable; or (b) there is a firm j such that $\overline{\Pi}_j$ is strictly finer than Π^0_j. The former case resembles Roth and Vande Vate (1990) (henceforth, RV), and the latter is specific to the incomplete-information setting.

Lemma 2. *Let A^0 be a set of agents which is internally stable under a state $(\mu^0, \mathbf{p}^0, \Pi^0)$. Suppose that worker $i^0 \notin A^0$ (resp. firm $j^0 \notin A^0$) forms a blocking pair for $(\mu^0, \mathbf{p}^0, \Pi^0)$ with a firm (resp. a worker) in A^0. Then, starting from state $(\mu^0, \mathbf{p}^0, \Pi^0)$, there exists a finite learning-blocking path to a state $(\overline{\mu}, \overline{\mathbf{p}}, \overline{\Pi})$ under which either (a) the internally stable set is expanded, i.e., $A^0 \cup \{i^0\}$ (resp. $A^0 \cup \{j^0\}$) is internally stable; or (b) there exists a firm j whose information partition is strictly refined, i.e., $\overline{\Pi}_j$ is strictly finer than Π^0_j.*

To prove Lemma 2, we will explicitly construct a finite learning-blocking path which either (a) outputs a state such that the internally stable set is expanded in the sense of set inclusion; or (b) identifies a coalition (i, j, p), satisfying which leads to a strictly finer partition for firm j.[1] The lemma is proved using what we call the Worker-Adding (Firm-Adding) Algorithm. The algorithm resembles the argument of RV by trying to expand the internally stable set but differs in an essential manner, i.e., the internally stable set may shrink to an empty set due to strict information updating.

Proof of Lemma 2. Consider the case of a worker $i^0 \notin A^0$ who forms a blocking pair for state $(\mu^0, \mathbf{p}^0, \Pi^0)$ with a firm in A^0. To prove the claim, we input state $(\mu, \mathbf{p}, \Pi) =$

[1] Lemma 3 of CH20 is not used, but a modified idea is used in the last step of the current proof.

$(\mu^0, \mathbf{p}^0, \Pi^0)$, $A = A^0$, worker $\alpha = i^0$, and $(i, j; p) = (\emptyset, \emptyset; 0)$ into the following Worker-Adding Algorithm. The case of a firm $j^0 \notin A^0$ who forms a blocking pair with a worker in A^0 can be similarly proved by switching the roles of worker and firm. In describing the algorithm, for each input value x of a variable, we denote by x' the updated value of x.

The Worker-Adding Algorithm

Start. Input state $(\mu, \mathbf{p}, \Pi)$, a (possibly empty) subset A of $I \cup J$, worker α, and $(i, j; p)$ where $i \neq \alpha$. Consider four mutually exclusive cases:

Case 1. *There exists a blocking coalition for* $(\mu, \mathbf{p}, \Pi)$ *which includes worker* α *and some firm in* A.

Pick an arbitrary blocking coalition $(\alpha, \bar{j}; \bar{p})$ with firm $\bar{j} \in A$ for $(\mu, \mathbf{p}, \Pi)$. Consider two mutually exclusive cases:

Subcase 1.1. $(i, j; p)$ *constitutes a blocking coalition for* $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$.

Set $(\mu', \mathbf{p}', \Pi') = \left((\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}\right) \uparrow_{(i,j;p)}$, $A' = A \cup \{\alpha\}$, worker $\alpha' = \alpha$, and $(i', j'; p') = (\mu^{-1}(\bar{j}), \bar{j}; \mathbf{p}_{\mu^{-1}(\bar{j}), \bar{j}})$. Go to Start.

Subcase 1.2. $(i, j; p)$ *does not constitute a blocking coalition for* $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$.

Set $(\mu', \mathbf{p}', \Pi') = (\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$ and $A' = \varnothing$. Go to end.

Case 2. *There exists no blocking coalition for* $(\mu, \mathbf{p}, \Pi)$ *which includes worker* α *and a firm in* A *but there exists a blocking coalition for* $(\mu, \mathbf{p}, \Pi)$ *which includes worker* i *and some firm in* A.

Pick an arbitrary blocking coalition $(i, \bar{j}; \bar{p})$ with firm $\bar{j} \in A$ for $(\mu, \mathbf{p}, \Pi)$. Set $(\mu', \mathbf{p}', \Pi') = (\mu, \mathbf{p}, \Pi) \uparrow_{(i, \bar{j}; \bar{p})}$, $A' = A$, worker $\alpha' = i$, and $(i', j'; p') = (\mu^{-1}(\bar{j}), \bar{j}; \mathbf{p}_{\mu^{-1}(\bar{j}), \bar{j}})$. Go to Start.

Case 3. *There exists no blocking coalition for* $(\mu, \mathbf{p}, \Pi)$ *which includes either worker* α *or worker* i *and a firm in* A. *However, there exists a blocking coalition* $(\bar{i}, \bar{j}; \bar{p})$ for $(\mu, \mathbf{p}, \Pi)$ *with both the firm and the worker in* A.

Set $(\mu', \mathbf{p}', \Pi') = (\mu, \mathbf{p}, \Pi) \uparrow_{(\bar{i}, \bar{j}; \bar{p})}$ and $A' = \varnothing$. Go to end.

Case 4. *There exists no blocking coalition for* $(\mu, \mathbf{p}, \Pi)$ *which includes a pair of agents in* $A \cup \{\alpha\}$.

Set $(\mu', \mathbf{p}', \Pi') = (\mu, \mathbf{p}, \Pi)$ and $A' = A$. Go to END.

END. Output $\overline{A} := A'$ and $(\overline{\mu}, \overline{\mathbf{p}}, \overline{\Pi}) := (\mu', \mathbf{p}', \Pi')$.

We proceed to explain the algorithm, and then complete the detailed arguments.

The algorithm keeps track of the following variables to be updated in each step: a state $(\mu, \mathbf{p}, \Pi)$, a set A of agents, a worker α, and a "potential blocking coalition $(i, j; p)$." In each step, exactly one of the four cases will be triggered. Case 1 says that worker α has the first priority to block the state: as long as he still wants to do so, we update the state by satisfying one such blocking coalition, say matching worker α and firm $\bar{j}$. At the same time, we check whether the potential blocking coalition $(i, j; p)$ is indeed a blocking coalition for the state updated from satisfying α. If that is the case, we satisfy $(i, j; p)$, update A to $A \cup \{\alpha\}$, and update the "potential blocking coalition" to $(i', j'; p')$ where j' becomes firm $\bar{j}$, and i' and p' become the employee of firm $\bar{j}$ and the wage which firm $\bar{j}$ paid to him before she left for worker α, respectively. If not, the algorithm terminates and outputs the updated state and $\varnothing$. Hence, when Case 1 is triggered at a state in which firm j and worker α are matched, the "potential blocking coalition" $(i, j; p)$ is actually firm j's match immediately before she is matched with worker α.

Unless Subcase 1.2 is triggered, in which case we identify a strict refinement of firm j's partition (to be proved below), we keep triggering Subcase 1.1 until there is no more blocking coalition which involves worker α. Then, we turn to trigger Case 2 if there is a blocking coalition which involves worker i. In this case, we update the state by satisfying one such blocking coalition, say matching worker i and firm $\bar{j}$; moreover, we let worker i become the new worker α and update the "potential blocking coalition" to $(i', j'; p')$ where j' becomes firm $\bar{j}$, and i' and p' become the employee of firm $\bar{j}$ and the wage which firm $\bar{j}$ paid to him before she left for worker α, respectively. In addition to Subcase 1.2, the algorithm will stop once there is no blocking coalition which involves either worker α or worker i. Then, the algorithm outputs the updated state and $\varnothing$ if there is still a blocking coalition with two agents both in A; otherwise, it outputs the state and the set A of the final step.

First of all, we claim that the algorithm produces a learning-blocking path. This is obviously true because along the algorithm, the state is updated only by satisfying blocking coalitions.

Secondly, we claim that whenever Subcase 1.2 is triggered, firm j's partition must be strictly refined. Suppose to the contrary that firm j's partition has never been refined along the learning-blocking path starting from $(\mu^0, \mathbf{p}^0, \Pi^0)$. Then we proceed to argue that $(i, j; p)$ must be a blocking coalition for $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$, which is a contradiction. To see this, we consider two possibilities. For the initial input $(i, j; p) = (\emptyset, \emptyset; 0)$ or the notational case $(i, j; p) = (\emptyset, j; 0)$, $(i, j; p)$ being a blocking coalition is a dummy condition. If $i \in I$ and $(i, j; p)$ is updated after either Subcase 1.1 or Case 2 is triggered, worker i is unmatched under $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$; moreover, firm j is also unmatched under $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$ since she has just been dumped by α. Since every matched agent in A has a strictly positive payoff (and remains so along the path as long as her/his information is not updated), we know that i and j both prefer to be rematched with each other at p rather than standing alone, i.e., $(i, j; p)$ must be a blocking coalition for $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$. This contradicts the underlying Subcase 1.2, so firm j's partition under $(\mu, \mathbf{p}, \Pi) \uparrow_{(\alpha, \bar{j}; \bar{p})}$ has to be strictly finer than Π^0_j.

Thirdly, we claim that the constructed path is finite. Since strict information refinement can happen only finitely many times, without loss of generality, we suppose there is no strict information updating in this paragraph. By discreteness of payments, some firm's payoff is strictly increased when Subcase 1.1 or Case 2 is triggered; moreover, a firm's payoff never decrease in the algorithm unless she is firm j in Subcase 1.1, in which case her payoff drops after a temporary increase and, at the same time, some other firm's payoff must strictly increase. Since we have only finitely many firms in A, the constructed path can be infinite only if some firm's payoff is improved indefinitely. However, this contradicts individual rationality of workers and that the surpluses are uniformly bounded across all matches.

Finally, if the path terminates by triggering Case 4, then $\overline{A} = A^0 \cup \{i^0\}$ is internally stable.[2] Hence, it suffices to argue that when the path terminates by triggering Case 3, the updated partition of the blocking firm $\bar{j}$ in Case 3 must be strictly finer than her partition under the initial state. To see this, we denote by $\left\{\left(\alpha^k, i^k\right)\right\}_{k=1}^K$ the sequence of workers who serve in order the role of worker α and the role of worker i in Case 2 up until Case 3 is triggered. Let $k^* \leq K-1$ be the maximal k such that $\bar{i} = \alpha^k$ ($k^* := 0$ if $\bar{i}$ has never served the role of worker α). Hence, worker $\bar{i}$ is neither α^k nor i^k for every $k \geq k^*+1$.

Since only worker α or worker i changes his partner or wage in the algorithm before it terminates, worker $\bar{i}$ remains matched with the firm and wage which he settled as worker α^{k^*} (or at the initial state $(\mu^0, \mathbf{p}^0, \Pi^0)$ if $k^* = 0$) ; denote that particular state $(\mu, \mathbf{p}, \Pi)$ in the underlying Case 2 by $(\mu^*, \mathbf{p}^*, \Pi^*)$. Case 2 must be triggered when worker α^{k^*+1} succeeds worker $\bar{i}$ to become a new worker α and when there was no more blocking coalition which involves worker $\bar{i}$. In particular, $(\bar{i}, \bar{j}; \bar{p})$ was *not* a blocking coalition for $(\mu^*, \mathbf{p}^*, \Pi^*)$. Recall that the blocking conditions of $(\bar{i}, \bar{j}; \bar{p})$ for $(\mu^*, \mathbf{p}^*, \Pi^*)$ are:

$$a_{\bar{i}\bar{j}}(t) + \bar{p} \;>\; a_{\bar{i}\mu^*(\bar{i})}(t) + p^*_{\bar{i}\mu^*(\bar{i})} \quad \text{and} \tag{1}$$

$$\mathbb{E}_{\beta^0}\left[b_{\bar{i}\bar{j}} \middle| \Pi^*_{\bar{j}}(t) \cap D^*_c\right] - \bar{p} \;>\; \mathbb{E}_{\beta^0}\left[b_{\mu^*(\bar{j})\bar{j}} \middle| \Pi^*_{\bar{j}}(t) \cap D^*_c\right] - p^*_{\mu^*(\bar{j})\bar{j}}, \tag{2}$$

where

$$D^*_c \;:=\; \left\{t' \in T: \quad a_{\bar{i}\bar{j}}(t') + \bar{p} > a_{\bar{i}\mu^*(\bar{i})}(t') + p^*_{\bar{i}\mu^*(\bar{i})}\right\}. \tag{3}$$

Since $(\bar{i}, \bar{j}; \bar{p})$ is *not* a blocking coalition for state $(\mu^*, \mathbf{p}^*, \Pi^*)$, either (1) or (2-3) fails.

Denote the state in Case 3 which admits the blocking coalition $(\bar{i}, \bar{j}; \bar{p})$ by

[2] In case the output state admits some *matched* agent in $\overline{A}$ having an expected payoff of zero, as long as the starting point is internally stable (every matched agent in A has a strictly positive expected payoff), this zero matched payoff must be a result of strict information updating, fitting into case (b) of the lemma. Thus, it is without loss of generality to assume that all matched agents in $\overline{A}$ have strictly positive expected payoffs.

$(\mu^\dagger, \mathbf{p}^\dagger, \Pi^\dagger)$. The blocking conditions of $(\bar{i}, \bar{j}; \bar{p})$ for $(\mu^\dagger, \mathbf{p}^\dagger, \Pi^\dagger)$ are:

$$
\begin{aligned}
a_{\bar{i}\bar{j}}(t) + \bar{p} &> a_{\bar{i}\mu^\dagger(\bar{i})}(t) + p^\dagger_{\bar{i}\mu^\dagger(\bar{i})} \quad \text{and} & (4)\\
\mathbb{E}_{\beta^0}\left[b_{\bar{i}\bar{j}} \middle| \Pi^\dagger_{\bar{j}}(t) \cap D^\dagger_c\right] - \bar{p} &> \mathbb{E}_{\beta^0}\left[b_{\mu^\dagger(\bar{j})\bar{j}} \middle| \Pi^\dagger_{\bar{j}}(t) \cap D^\dagger_c\right] - p^\dagger_{\mu^\dagger(\bar{j})\bar{j}}, & (5)
\end{aligned}
$$

where

$$
D^\dagger_c \;:=\; \left\{ t' \in T : \quad a_{\bar{i}\bar{j}}(t') + \bar{p} > a_{\bar{i}\mu^\dagger(\bar{i})}(t') + p^\dagger_{\bar{i}\mu^\dagger(\bar{i})} \right\}. \tag{6}
$$

Since worker $\bar{i}$ remains matched as at $(\mu^*, \mathbf{p}^*, \Pi^*)$ (where he settled as worker α^{k^*}) even when the state becomes $(\mu^\dagger, \mathbf{p}^\dagger, \Pi^\dagger)$, condition (4) is identical to (1), and $D^\dagger_c$ identical to D^*_c. Then $(\bar{i}, \bar{j}; \bar{p})$ *being* a blocking coalition for $(\mu^\dagger, \mathbf{p}^\dagger, \Pi^\dagger)$ but not $(\mu^*, \mathbf{p}^*, \Pi^*)$ implies that condition (5-6) holds but (2-3) fails.

If $\Pi^\dagger_{\bar{j}}$ is already strictly finer than $\Pi^*_{\bar{j}}$, we are done. Now suppose $\Pi^\dagger_{\bar{j}} = \Pi^*_{\bar{j}}$. Then $\Pi^\dagger_{\bar{j}}(t) \cap D^\dagger_c = \Pi^*_{\bar{j}} \cap D^*_c$ and the left-hand side of (5) is identical to that of (2). A further implication of $\Pi^\dagger_{\bar{j}} = \Pi^*_{\bar{j}}$ is that in all coalitional deviations that firm $\bar{j}$ experienced from $(\mu^*, \mathbf{p}^*, \Pi^*)$ to $(\mu^\dagger, \mathbf{p}^\dagger, \Pi^\dagger)$, if any, none of those blocking workers' willingness (or any other observations) has provided any information to help firm j refine her partition. We claim that $\Pi^\dagger_{\bar{j}}(t) \cap D^\dagger_c \neq \Pi^*_{\bar{j}}$. Suppose to the contrary that they are the same. Then the right-hand sides of (5) and (2) can be compared as follows:

$$
\mathbb{E}_{\beta^0}\left[b_{\mu^\dagger(\bar{j})\bar{j}} \middle| \Pi^\dagger_{\bar{j}}(t)\right] - p^\dagger_{\mu^\dagger(\bar{j})\bar{j}} \geq \mathbb{E}_{\beta^0}\left[b_{\mu^*(\bar{j})\bar{j}} \middle| \Pi^*_{\bar{j}}(t)\right] - p^*_{\mu^*(\bar{j})\bar{j}},
$$

since firm $\bar{j}$ always obtains a higher expected payoff via coalitional deviations. Therefore, condition (5-6) is harder to satisfy than (2-3), which contradicts to the fact that condition (5-6) holds but (2-3) fails. Hence, we must have $\Pi^\dagger_{\bar{j}}(t) \cap D^\dagger_c \neq \Pi^*_{\bar{j}}$.

Consequently, satisfying $(\bar{i}, \bar{j}; \bar{p})$ in Case 3 could refine firm $\bar{j}$'s information, in terms of partition cell, from $\Pi^*_{\bar{j}}(t)$ to $\Pi^\dagger_{\bar{j}}(t) \cap D^\dagger_c$. (In other words, satisfying the coalition turns the hypothetical information of worker $\bar{i}$'s willingness, represented by $D^\dagger_c$, into a new piece of real information for firm $\bar{j}$.) $\square$

Proof of the Proposition. Consider an initial state. Without loss of generality, we assume that the initial state is individually rational (otherwise, break up all pairs with an agent who obtains a negative payoff). Furthermore, it is straightforward that any learning-blocking path preserves individual rationality, unless there is strict information updating. Since it is impossible to have strict information updating indefinitely, it is without loss to assume that individually rational is always satisfied along the constructed learning-blocking path.

We take an initial set of agents $A^0 = \varnothing$, where the variable A will be updated during the construction below. For each state $(\mu, \mathbf{p}, \Pi)$, we distinguish two cases: (i) $(\mu, \mathbf{p}, \Pi)$ admits no blocking pair; (ii) $(\mu, \mathbf{p}, \Pi)$ admits a blocking pair. First, by Lemma 1, for each state $(\mu, \mathbf{p}, \Pi)$ in Case (i), there is a finite learning-blocking path to either a stable state or a state in Case (ii) with a partition profile that is strictly finer than Π, where we update A to $A' = \varnothing$ and proceed. Second, for each state $(\mu, \mathbf{p}, \Pi)$ in Case (ii) with a set of agents A which is internally stable under $(\mu, \mathbf{p}, \Pi)$, consider two subcases: Firstly, if some blocking coalition involves an agent in A and an agent outside A, then by Lemma 2, starting from $(\mu, \mathbf{p}, \Pi)$, there exists a finite learning-blocking path to a state $(\mu', \mathbf{p}', \Pi')$ under which either (a) a set $A' \supsetneq A$ is internally stable; or (b) Π' is strictly finer than Π, where we set $A' = \varnothing$ and proceed. Secondly, if every blocking coalition involves only two agents outside A, then we satisfy a blocking coalition $(\bar{i}, \bar{j}; \bar{p})$ to obtain $(\mu, \mathbf{p}, \Pi) \uparrow_{(\bar{i}, \bar{j}; \bar{p})}$. If $A' = A \cup \{\bar{i}, \bar{j}\}$ is internally stable under $(\mu, \mathbf{p}, \Pi) \uparrow_{(\bar{i}, \bar{j}; \bar{p})}$, then (a) holds. If $A' = A \cup \{\bar{i}, \bar{j}\}$ contains a blocking pair (i, j), with wage p, for $(\mu, \mathbf{p}, \Pi) \uparrow_{(\bar{i}, \bar{j}; \bar{p})}$, then we can use the same argument as in the proof of Lemma 2 (particularly how Case 3 identifies a strict information updating) to see that (b) holds for $(\mu', \mathbf{p}', \Pi') := \big((\mu, \mathbf{p}, \Pi) \uparrow_{(\bar{i}, \bar{j}; \bar{p})}\big) \uparrow_{(i, j; p)}$, where we set $A' = \varnothing$ and proceed.

To sum up, for each state in Case (ii) with a set A which is internally stable, we can construct a finite learning-blocking path to a state under which either (a) a set $A' \supsetneq A$ is internally stable; or (b) the partition profile is strictly finer. Moreover, for each state in Case (i), we can also construct a finite learning-blocking path to either (a′) a stable state or (b′) a state in Case (ii) with a strictly finer partition profile. Since T is finite, the partition profile cannot be refined indefinitely. Hence, along the path which we construct by applying Lemma 1 and Lemma 2, eventually neither (b) nor (b′) happens. Therefore, (a) keeps enlarging the internally stable set until (a′) happens.

That is, there is a finite learning-blocking path which leads to a stable state. □

References

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